Dear reviewer, we sincerely appreciate your thorough review that can improve the manuscript. Please find our responses below. Your comments are highlighted in blue.

Reviewer comment: I read with great interest this manuscript which reports about the attempt at describing the fingering process in a two dimensional porous medium by means of a semicontinuum approach. The article is rich in the literature and in the analyses. As a general recommendation I suggest:

1. To describe with some more details the background assumptions and the model;
2. To perform a more detailed comparison (even quantitative, if possible) between previous experiments and the numerical findings

In the followings I detail a little more the questions arised from the reading of the paper, that I recommend to address in the review.

- Thank you. Since these general recommendations are discussed in more detail below, we have chosen to comment on them there.

Reviewer comment: The Authors represent a soil section with three main assumptions:

1. The SWRC is represented by a mixed form of a classical van Genuchten's SWRC (in an hysteretic form) with a minimal Prandtl's hysteresis operator. The merge of the two curves is ruled by the dimension of the simulation cell. This idea is close to that the experimental SWRCs are sensitive to the dimension of the laboratory sample, being more flattened as soon as the sample dimension is reduced. In this sense the Prandtl operator is seen as the minimal behaviour of a single capillary. Then they fix the cell dimension and accordingly a SWRC for the soil is found.

   I recommend to better evidence the reason for the choice of the cell dimension, and to express with more detail whether the simulations are expected to be sensitive to the cell dimension (as the SWRC seems conditioned to it).

- The reference block size Δx_0 of the 20/30 sand was calibrated in [1] by simulating experiments conducted by Bauters et al. [2]. In [1], we ran several simulations of the semi-continuum model using three different values of Δx_0, and we calculated the moisture profile for three different initial saturations: a dry, a medium dry, and a wet porous medium. We selected $\Delta x_0 = \frac{10}{12}$ as the most appropriate option. Further details of the calibration process are provided in Section 3.1 in [1].

  Here, we used the same parameters of 20/30 sand as given in [1]. This includes parameters such as the van Genuchten's parameters of the retention curve, relative permeability exponent, intrinsic permeability, and the reference block size. Hence, our aim was to use these identical parameters for different flow phenomena, thus avoiding parameter fitting to achieve the best possible agreement with the experiments.

- Since the retention curve is sensitive to the dimension of the laboratory sample, this influences simulations that are sensitive to the size of the reference block Δx_0. This poses a minor limitation of the semi-continuum model because its objective is to model flow phenomena below the reference elementary volume, where the dependency on the sample volume is significant. However, once the reference block size Δx_0 is appropriately set, simulations are no longer sensitive to the specific block size Δx used. Therefore, the results are independent of the size of the elements used for simulations, which is a crucial characteristic for any numerical model. For example, we refer to Figures 4–6 in [3], where the convergence of the moisture profile in 1D/2D is demonstrated for the block size varying over almost two orders of magnitude.

- We plan to include an explanation for the choice of the reference block size Δx_0 to clarify that our aim was not to fit the results. Moreover, the results are not sensitive to the block size Δx, while this does not apply to the reference block size Δx_0, which is a parameter of the semi-continuum model.

2. The intrinsic permeability is represented by means of a stocastic simulation. It is really needed? Probably it seems not, as the Authors report in the discussion, but it anyway could mix the effects of hysteresis with the effect of the permeability field on affecting the formation of preferential pathways. Have the Authors considered the possibility of performing separate simulations to evidence the relative importance of hysteresis vs permeability field?

- Since the reviewer repeatedly raised questions about the importance of intrinsic permeability distribution, we decided to comprehensively address all questions at this point. Subsequently, we will extend our explanation in the manuscript of why we used a distribution of intrinsic permeability and what governs the formation of the saturation overshoot.

- The saturation overshoot is formed by two factors: (1) the geometric mean for averaging the permeability and (2) the dependency of the retention curve on the block size, the so-called the scaling of the retention curve.

  The geometric mean plays a crucial role in creating the hold-back effect in the semi-continuum model. This effect occurs due to the the very low relative permeability between the blocks, which is a direct consequence of the applied geometric mean. However, without scaling of the retention curve, the overshoot would disappear as the block size $\Delta x \to 0$. This has already been discussed in [3]; we refer to Figure 2 in [3], where we demonstrated that the saturation overshoot disappears when the scaling of the retention curve is not included. Let us note that the geometric mean can be replaced by any type of averaging that satisfies the crucial property of being small if one of the averaged numbers is small. Hence, it is also possible to use, for instance, the harmonic mean. We chose the geometric mean, as it is more appropriate in the case of a random stratified medium [4], while the harmonic mean is more suitable for a perpendicular stratified medium.

- The formation of the saturation overshoot is not influenced by the distribution of the intrinsic permeability. This is evidenced by 1D simulations; the saturation overshoot is formed even when the distribution is not used, as shown in Figure 10 in the manuscript. Moreover, this corresponds with findings from experimental observations [5], where the formation of saturation overshoot is not determined by heterogeneity. Hence, the question arises: Why use the distribution of the intrinsic permeability if it does not affect the formation of the saturation overshoot? Because a slight distribution of the intrinsic permeability will cause the water to not flow uniformly through the entire porous medium in 2D/3D. This is expected because no heterogeneity is introduced in the governing equation; hence, water will always flow uniformly unless some additional heterogeneity is introduced. This intrinsic permeability distribution is typical for similar models used for unsaturated porous media flow [6]. Another possibility is, for instance, using initial saturation with small perturbations near the top boundary, as used in [7], or introducing small perturbations in a top boundary condition. We plan to include separate simulations in the Appendix without intrinsic permeability distribution to demonstrate that the used distribution is not the cause of the formation of the saturation overshoot in 2D, but rather causes water to flow preferentially.

  To summarize: (1) When using only the distribution of the intrinsic permeability without incorporating the geometric mean and the scaling of the retention curve, the overshoot will not be formed. Consequently, the flow behavior is diffusive in this scenario and water flows throughout the entire porous medium. (2) When using only the geometric mean and the scaling of the retention curve, the overshoot can be formed (depending on initial and boundary conditions). However, the flow remains uniform throughout the entire porous medium. By combining (1) and (2), the saturation overshoot is formed and water does not flow uniformly.

- Moreover, when the distribution is employed but the saturation overshoot does not occur (e.g., in the case of low influx), water tends to flow diffusively in the semi-continuum model. This aligns well with experimental observations. Other similar models fail in this case, as water flows preferentially even when the saturation overshoot is not formed. In our case, if the overshoot is formed, preferential flow is observed. If there is no overshoot, preferential flow disappears. The distribution of the intrinsic permeability is included to make water flow non-uniformly (preferentially) throughout the entire porous medium, but this non-uniformity occurs only when the physics of the semi-continuum model allows it, i.e., when the overshoot is formed. And this applies to any heterogeneity that can be included in the model, such as small perturbations in the initial and boundary conditions.

3. The relative permeability is represented by means of a classical power law / or van Genuchten Mualem shape.

With these hypotheses, it is assumed that the conservation of mass with the Darcy–Buckingham law, admits the formation of a saturation overshoot in the 1d form, and the formation of front instability in the 2d form (as it does in the simulations). The obtained equation is in my opinion a form of the Richards equation with hyteretic SWRC, because, as far as the soil is discontinuous and we aim at representing its properties by means of descriptive index properties (as the saturation or the porosity are), we implicitly admit that the Richards equation is locally defined by means of average values on a certain small domain, which is commonly referred to as the Representative Elementary Volume – even if not defining its dimension. What moves the present model from the Richards equation is the choice of the dimension of the cell, which rules the behaviour of the SWRC.

- While it is true that the semi-continuum model is similar to the Richards' equation, it is important to note, as pointed out by the reviewer, that the block size rules the behavior of the retention curve. This makes the formal limit of the semi-continuum model mathematically significantly different. Richards' equation is a parabolic differential equation, whereas the formal limit of the semi-continuum model is a parabolic-hyperbolic equation.

Reviewer comment: Regarding the formation of an overshoot the Authors refer together to two kinds of overshoots which are very different in their meaning. In fact an overshoot related to a pulsation in the Dirichlet boundary condition is in any case in agreement with the parabolic behaviour of the Richards equation, as it can be in any case framed within the maximum principle which guarantees the solution of the parabolic operators. Consider, for example, the Stokes problem of the velocity profile in a seminifinte steady fluid with pulsating wall.

- We agree with the reviewer. For instance, a time-dependent Dirichlet boundary condition could indeed form saturation overshoot in the parabolic differential equation [8]. However, since the experiments we are replicating utilized a monotonic Dirichlet boundary condition, any pulsations in the boundary condition are of no interest. In this case, the Richards' equation with a smooth and non-decreasing retention curve is unconditionally stable [9]. We will clarify this point in the manuscript.

Reviewer comment: Another is the case for which the maximum principle is not proven, as it can happen in some cases in which the soil is not homogeneous (see Barontini et al, 2007, WRR). This is the intriguing case which, according to the Authors, may lead to the instability. Is it possible to check the applicability of the maximum principle for the investigated case? I mean: the maximum principle is unprovable in the equation? or it is provable in the equation but it can be put into discussion as a consequence of the unhomogeneities of the soil permeability?

- As previously discussed, soil permeability inhomogeneities are not the cause of the saturation overshoot. The heterogeneity of the porous medium is neither a necessary nor a sufficient condition for the formation of instabilities.

  Moreover, we did not perform an analysis regarding the maximum principle. The saturation overshoot is not conditional on soil permeability inhomogeneities, hence the maximum principle cannot be proven. It would be interesting to attempt to prove otherwise; that the solution of the semi-continuum model is unstable. However, the formal limit of the semi-continuum model is a rather complex mathematical object, and we are not aware of any research addressing equations of this type. Therefore, such an analysis would not be straightforward. Anyway, we appreciate the suggestion for potential future research.

Reviewer comment: The results are interesting, particularly as the model describes a concentration of flow in the most saturated soil, even in the case in which the fingers are not developed. This is in agreement with the fact that the relative conductivity drops down as soon as the soil is not completely saturated. This behaviour is most evident in organic soils, where van Gencuchten's n is small (smaller than 2), but it is evident in any porous medium, also in this case for n between 6 and 8.

- This occurs for an intermediate case between the diffusion and finger-like flow, with the fingers not fully developed. This output from the the semi-continuum is interesting for us as well, as it provides detailed insight into the transition from diffusion to finger-like flow.

Reviewer comment: Yet one may argue whether the results account for a physical behavious or for a mathematical description intrinsic to the model. This is why I recommend to the Authors (1) to better focus on which is in their opinion the physical source fo the instaibility, whether it is the hysteresis or the variability of the permeability. (in this case it cannot be the presence of macropores, as – if I properly understand – macropores are not describerd in the model) and (2) to provide closer comparison with literature experimental results.

- Regarding the physical source of the instability, we refer to our previous responses to avoid repetition. Additionally, it is worth noting that macropores are indeed not included in the model.

- Regarding a closer comparison with experimental results from the literature, we have already performed several comparisons in 1D and 2D. Specifically, we refer to [10] for 1D simulations and to [11, 1] for 2D simulations. For instance, in [1], we demonstrated that the semi-continuum model is capable to fully reproduce the transition from finger-like flow in an initially dry medium do diffusion-like flow in an initially wet medium for a point source infiltration.

We plan to better explain our motivation in the manuscript, and why we chose to simulate 2D infiltration experiments of Yao and Hendrickx [12] and Glass et al. [13]. Our goal is to fully validate the semi-continuum model with well-known laboratory experiments, following DiCarlo's approach [5]. DiCarlo suggested evaluating a model, which consists of four points (see section 6.6 in [5]). The only point the semi-continuum model has not fully addressed is the fourth one, which states that the model "Can produce predictions of the 2-D and 3-D preferential flow in terms of finger widths and finger spacings." We have already achieved successful reproduction of the dependency on initial saturation [11, 1]. However, the final aspect required for full validation is to accurately capture the dependence on the infiltration rate of 2D/3D experiments [12, 13, 5]. Furthermore, ongoing efforts within the community, as evidenced by Beljadid et al. [7], emphasize the continued interest in reproducing these experiments.

Reviewer comment: Before closing, I add some minimal notes: The Authors express the conductivity of the cell as the geometric average of two conductivities at different water content (the minimum and the maximum, it seems, but it shoud be probably better enlightened). Where does this scheme come from?

- We do not use either minimum or maximum of the geometric mean. The misunderstanding likely arises from the notation $k(S^-)$ and $k(S^+)$ in Eq. (1b). These denote the left and right limits, as described in Eq. (1b), i.e., $S^\pm(x_0, t) = \lim_{x \to x_0^\pm} S(x, t)$.

  Notation in Eq. (1b) may be confusing, and we intend to improve it. The confusion likely comes from the presence of discontinuous saturation, which arises from the use of a geometric mean conductivity and subsequent limiting process [3]. Due to the complexity caused by the use of a Prandtl-type hysteresis operator, we were unable to prove whether the saturation remains continuous in the limit or not. Therefore, for an accurate mathematical description, it is necessary to preserve the discontinuity when using the geometric mean for conductivity.

- We use a geometric mean of conductivity for two adjacent blocks, as described in Eq. (6) in the manuscript. This type of averaging is consistent with the work of Jang et al. [4]. Note that the geometric mean has a desirable property of being small if one of the conductivities is small, which does not apply for the more standard arithmetic mean. This has several practical implications; for instance, it eliminates the necessity to implement the water-entry value in the retention curve, as the geometric mean ensures nearly zero flux for very small saturations. This feature has already been discussed here: `doi.org/10.5194/egusphere-2023-2785-AC3`, and we plan to incorporate this information into the manuscript.

Reviewer comment: Mualem's parameter ℓ is usually set at 0.5 (despite it can be changed, if needed): why did the Authors set at 0.8? Does it come form a fit of an experimetnal conductivity curve?

- We did not fit any experimental conductivity curve but used the same parameters as in [1], including the relative permeability exponent λ. This decision was made to demonstrate that the model is capable of simulating two different flow phenomena without the need for additional parameter adjustment. Our aim was not to optimize the parameters to achieve the best agreement with experiments, although the agreement is already very good.

  However, it's worth noting that the value of the parameter λ is consistent with measurements [14]. We intend to clarify this point in the manuscript.

Reviewer comment: Figure 1 is not very clear, I recomment to add the direction of the potential axis

- We agree that Figure 1 is indeed not clear. The same issue was raised by Reviewer n.2, so we have included a part of our response below for clarity.

- The reader is typically not familiar with a partial differential inequality, which is characterized in the model by a Prandtl-type hysteresis operator. To improve clarity, we have chosen to modify Figure 1 in the manuscript to better illustrate the Prandtl-type hysteresis model. Moreover, given that the Prandtl-type hysteresis operator provided by Eq. (1a) is not widely used in the soil science community, we plan to explain it before introducing the governing equation. Please find the modified Figure 1 of the Prandtl-type hysteresis operator below. In this figure, blue lines represent the limits of main wetting and draining branches, while black lines represent non-vertical scanning curves. The mathematical description of the Prandtl-type hysteresis operator can be found in Visintin [15] on page 16 (eqs. 2.3 - 2.5); see also the corresponding Figure 3 on page 15.

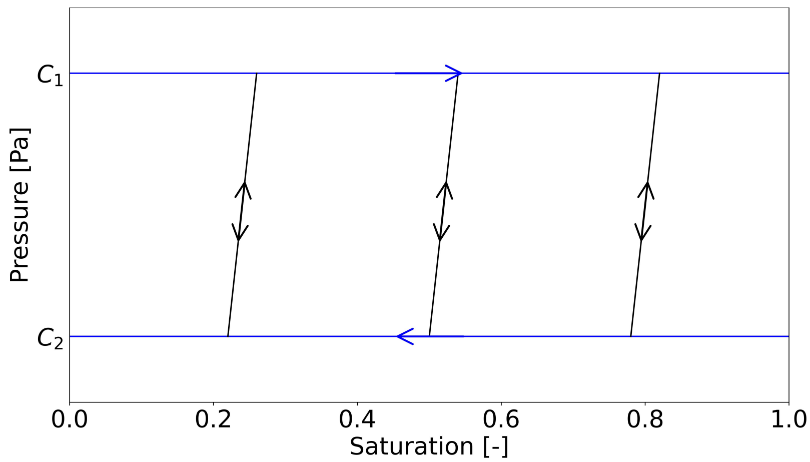

Figure 1: Prandtl-type hysteresis operator.

Reviewer comment: eq. 3: why 0.5?

- We chose to use the midpoint, but any arbitrary value from the interval $[0, 1]$ can be chosen, e.g., the inflection point. However, the effect on the results is negligible because the flux is calculated relative to the pressure gradient. We plan to clarify this in the manuscript.

Reviewer comment: l.226: please better detail the meaning of residual and initial saturation as residual is greater than initial

- The residual saturation refers to the maximum amount of water the porous medium can retain against the force of the gravity. The terminology is borrowed from Bauters et al. [2], where the authors filled the chamber of the 20/30 sand with water and then drained it to a residual water content of $0.047\,\mathrm{cm}^3/\mathrm{cm}^3$. Therefore, in our simulations, we have chosen the residual saturation to be 0.05. Initial saturation, on the other hand, refers to the saturation at the beginning of the simulation.

Reviewer comment: How the bottom boundary condition was chosen? If the column is suspended a seepage face condition would have been more realistic, yet it requires soil saturation before the leakage starts;

- Firstly, let us stress that the terminology used in the manuscript is incorrect. We describe the bottom boundary condition as a “free discharge”; however, free discharge actually refers to unit gradient flow at the lower boundary. We will fix this terminology issue in the manuscript.
- Secondly, we model the outflow of water into the air. There are various bottom boundary conditions that can be implemented. However, we chose a boundary condition that does not affect the flow above the boundary. Simply put, we wanted to ensure that the boundary condition did not propagate into the porous medium.

Reviewer comment: l.443: it is referred to stabilized flow as a percolation flow (i.e. with null spatial rate of change of the tensiometer–pressure potential), but due to the boundary conditions it might not be a pure percolation flow. Thank you for the attention.

- The reviewer n.2 raised a similar question, challenging our statetement: “The saturation and pressure in the finger tail are constant”, and refered to the work of Cho et al. [16]. We provide our response to the reviewer n.2 below.
- According to a comprehensive experimental work of DiCarlo [17], the saturation is constant at the finger tail for the uniform top boundary condition. For higher fluxes, slight oscillations are observed, however, DiCarlo explains that these are experimental artifacts related to light transmission variations near the end of the tubes. The constant pressure in the finger tail was also measured experimentally [18].

  For completeness, we will acknowledge studies that have shown the opposite, such as the work of Cho et al. We will also provide references to DiCarlo’s research to support our claim.

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