Review of "General Formulation For the Distribution Problem: Prognostic Assumed PDF Approach Based on The Maximum–Entropy Principle and The Liouville Equation" by Yano et al.

After careful assessment, I have decided to recommend this manuscript for publication after a minor revision.

Some of the points I previously raised have been addressed in the revised manuscript. A guide for applying the theory to more general forms of systems is now included in Sec. 5.3. The meaning of the output-constrained maximum-entropy principle is further elaborated. Additional details for clarification are provided throughout. All of which contributed to improving the manuscript's quality.

What is still missing is a more thorough evaluation of the error in the derived model, as such information is crucial to understand the methodology's reliability. I agree, however, with the authors that deriving and evaluating the error systematically would be complex. To give readers a rough idea of when and how the derived model performs accurately, a more in-depth discussion of the error using simple examples would be valuable. For this purpose, I request the authors to expand the discussion related to Fig.2 (see also Comment (6) below). Additionally, I would suggest exploring other simple but qualitatively different cases, such as those involving fixed points (steady solutions):

- $d\phi/dt = -\phi$. Here, $\phi = 0$ is a globally stable fixed point.
- $d\phi/dt = \phi(\phi - 1)$. Here, $\phi = 0$ is a stable fixed point, and $\phi = 1$ is an unstable fixed point.
- $d\phi/dt = \phi(\phi - 1)(2 - \phi)$. Here, $\phi = 0, 2$ are stable fixed points, and $\phi = 1$ is an unstable fixed point.

Some more comments are provided below.

Major Comments

1) **[request] P.4 ll.118–119 "... F designates all …"**
   It is not at all obvious to the readers that F can be space dependent. Please clarify this point here. I would explain F is a functional of $\phi(x, y, z)$.

2) **[comment] P.6 ll.139–140 "…, there is no closed analytical formula for reconstructing the original distribution from a given series of moments: …"**
   For the authors' information, I found an interesting paper that is closely relevant to this problem.
   > Chao Dang and Jun Xu, "Novel algorithm for reconstruction of a distribution by fitting its first-four statistical moments", Applied Mathematical Modelling, Volume 71, 2019, Pages 505-524, https://doi.org/10.1016/j.apm.2019.02.040.

3) **[question] P.23 Eq.(5.10b)**

I think we can solve the derived model in a slightly different way. Let us consider $\{\langle\sigma_l\rangle\}$, not $\{\lambda_l\}$, are the prognostic variables.

1. From $\{\langle\sigma_l\rangle(t)\}$, we can estimate the PDF $p(\phi,t)$ by using the maximum entropy principle.
2. Using the estimated $p(\phi,t)$, we can calculate the $\{\langle F_{\sigma_l}\rangle(t)\}$.
3. Using Eq.(5.10b), we can numerically calculate $\{\langle\sigma_l\rangle(t+\Delta t)\}$.

By repeating this procedure, we can numerically calculate the time evolution of weights $\{\langle\sigma_l\rangle(t)\}$. Isn't this easier than solving Eq. (5.8a)?

4) [comment] Sec.5.3 "Generalization to the PDE system (2.1)"

I appreciate that the authors added this section; we can now see that the generalization of the proposed theory is indeed straightforward. From the derivation provided in this section, I also feel that we may not need to bring the Liouville equation.

5) [comment] P.28, ll.787–790 "... the solution breaks down beyond this point …"

I think this behavior is reasonable. Please note that the solution of $d\phi/dt = \phi^n, \phi(0) \neq 0$ blows up in finite time if $n > 1$.

6) [question, request] Fig.2

Is Fig.2b correct? If I understand Eq. (5.17e) correctly, $\lambda_1(t)$ does not depend on n when $m = 1$.

For each $m = 0, 1, 2, 3$, please plot the true $\lambda_1(t) = 1/\langle\phi\rangle(t)$ and discuss which choice of n is the most accurate.

Minor Comments

7) [request] P.5 ll.139–140 "Furthermore, in the present study, the source term, F, is assumed to be deterministic."

This is not correct. Brownian motion is considered in Sec. 4.4. Please rephrase.

Typo

8) P.8 l.219 "... be be …"

9) P.11 l.314 "... to constraint …" -> "... to constrain …"

10) P.21 l.582 "... with by …"

11) P.22 l.622 "... $\partial^2\phi/\partial\phi^2$" -> "... $\partial^2 p/\partial\phi^2$"

12) P.26 l.730 "... questioned form"

References

Chao Dang and Jun Xu, “Novel algorithm for reconstruction of a distribution by fitting its first-four statistical moments”, Applied Mathematical Modelling, Volume 71, 2019, Pages 505-524,https://doi.org/10.1016/j.apm.2019.02.040.