Reply to CC3: egusphere-2023-216

‘Comment on egusphere-2023-216’, Valerio Lembo, 19 May 2023

[The reviewer’s input is in italic font, while our responses are in regular font.]

I appreciated the reading of manuscript ”Dynamical Systems, Algebraic Topology and the Climate Sciences”, a review article submitted for consideration on Nonlinear Processes in Geophysics by Michael Ghil and Denisse Sciamarella. The contribution is part of an invitation-only special issue, ”Perspectives on Climate Sciences: from historical developments to research frontiers”, meant to be a follow-up of a webinar series organized under the auspices of the European Geosciences Union between 2020 and 2021.

The review article is aimed at bringing together authors’ work and views on recent developments in dynamical systems, especially non-autonomous ones, and the usage of features of branched manifolds defined in algebraic topology for the characterization of an attractor’s behavior. The article is well written in most of its parts, and it is a very enjoyable reading, especially for those not familiar with the specific topics addressed. I wish to share here a few remarks, and also some views on how these ideas can be expanded and find applications in climate sciences.

We are grateful for your careful reading and for helping us improve this manuscript.

SPECIFIC COMMENTS

- ll. 146 and elsewhere: a non-expert introduction to branched manifolds is missing, and I think it would be very welcome, given the potentially broad audience to whom the contribution is directed. Not only a mathematical definition of these objects (some idea of that is given at ll. 537-540), but rather expanding on the potential advantage of adopting this approach in the field of dynamical systems would be maybe helpful;

This point has been also raised by the first reviewer. An introduction to the concept of branched manifold is given below. We also comment on how this definition has been tailored to suit the different aspects of the approach that we present.

In “How topology came to chaos,” Gilmore explains that metric and dynamical invariants do not provide a way to distinguish among the different types of chaotic attractors and that a tool of a different nature is needed to create a dictionary of processes and mechanisms underlying a chaotic system.

"Listening more closely to Poincaré, it was clear that this new tool ought to involve the periodic orbits 'in' a chaotic attractor. A chaotic trajectory winds around in phase space arbitrarily close to any unstable periodic orbit, so it ought to be possible to use segments of a chaotic trajectory as good approximations (surrogates) for UPOs. [...] It was clear that UPOs could also serve as the skeleton of the strange attractor."

While Gilmore, Lefranc and co-workers were "mulling over implementing a program based on building tables of linking numbers and/or relative rotation rates between trajectories, a better solution became available. Joan Birman and Robert Williams had shown that the dissipative nature of a flow in phase space allows projecting the points along the direction of the stable manifold by identifying all the points with the same future."

"Suppose we have a dissipative chaotic flow in three dimensions: there are three Lyapunov exponents ($\lambda_1 > 0$ for the unstable direction, $\lambda_2 = 0$ for the flow direction and $\lambda_3 < 0$ for the stable direction). The dissipative nature of the flow requires $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Then it is possible to project points in the phase space down in the direction of the stable manifold. This is done by identifying all the points with the same future via the relation

$$x \sim y \text{ iff } \lim_{t \longrightarrow +\infty} |x(t) - y(t)| = 0$$

where $x(t)$ is the future in phase space of the point $x = x(0)$ under the flow. This Birman-Williams identification effectively projects the flow down to a manifold almost everywhere, except at the points where the flow splits into branches heading towards distinct parts of phase space, or at the points where two branches are squeezed together. These mathematical structures were called branched manifolds."

A branched manifold can in fact be defined mathematically without reference to a flow, or to the Birman-Williams projection mentioned above. Definition (from Kinsey page 64). An n-dimensional manifold is a topological space such that every point has a neighborhood topologically equivalent to an n- dimensional open disc with center x and radius r. Such a manifold is said to be Hausdorff if and only if any two distinct points have disjoint neighborhoods.

The second condition is not satisfied precisely at the junction between branches, i.e., at the locations that describe stretching and squeezing of a flow in phase space. A branched manifold is therefore a manifold that is not required to fulfill the Hausdorff property.

We prefer this more general definition, instead of the one related to the Birman-Williams projection, for several reasons, including the possibility of extending the concept of branched manifold to the structure of instantaneous snapshots of random attractors. This mathematical definition of a branched manifold will also let us extend the procedure to cases in which the hypotheses of the Birman-Williams theorem – in which the dynamical system must be hyperbolic, three-dimensional, and dissipative – are not valid. In most geoscientific applications, for instance, uniform hyperbolicity does not apply.

As the topological structure of a branched manifold is closely related to the stretching and squeezing mechanisms that constitute the fingerprint of a certain chaotic attractor, its properties can be used to distinguish among different attractors. This is how the two-way correspondence between topology and dynamics can be justified. This correspondence remains valid in the case of four-dimensional semi-conservative systems [Charó et al, 2019; Charó et al, JFM, 2021], for which the hypotheses of the Birman-Williams theorem do not hold.

The terms "branched manifold" and "template" have often been used interchangeably. We do not consider them as synonyms, for technical reasons that become important in the development of the concept of templex. A branched manifold is just a particular type of manifold that can be reconstructed from a set of points in $\mathbb{R}^n$, by approximating subsets of points by cells, which are glued to form a cell complex. The dimension d of the cell complex coincides, by construction, with the local dimension of the branched manifold approximating the point cloud, but there is no restriction on the value of either n or d. Both values are computed directly from the dataset, using successive singular value decompositions.

Since the number of eigenvalues scales linearly with the number of points grouped in a cell, this number provides the value of d for the given cell, and this computation is carried out on matrices that contain the n coordinates of the points, without performing projections of any kind. These computations construct a cell complex from the point cloud without involving the flow. The information carried by the flow is not contained in the cell complex but will be contained in the digraph of the templex.

- Figure 2: the label is not very self-explanatory and should be better detailed;

These three-dimensional point clouds are obtained by integrating the Lorenz equations using coordinate transformations for some of the variables. The butterfly is deformed but the topological structure of the butterfly is maintained. The caption will be expanded.

- ll. 366-371: This paragraph seems to be missing a take-home message;

The paragraph reads: "The finite-dimensional definition above follows Charó et al. (2021b, Appendix A and references therein). In fact, both deterministic and stochastic versions of [time-dependent] forcing have been applied, for instance, by Chekroun et al. (2018) in the study of an infinite-dimensional, delay-differential equation model of the El Niño–Southern Oscillation (ENSO). The deterministic forcing corresponded to the purely periodic, seasonal changes in insolation, while the stochastic component represented the westerly ind bursts appearing in various ENSO models by F.-F. Jin and A. Timmermann (e.g., Timmermann and Jin, 2002); see also Chekroun et al. (2011, Sec. 4.3)." The take-home message is that there is great flexibility in the application of the concepts and methods of nonautonomous dynamical systems (NDS and RDS) theory to climate problems. This will be mentioned in the revised version; thank you.

- Figure 8: the label here is also a bit ambiguous, as the invariant measure nu is not defined anywhere in the text;

A full definition of invariant measures would occupy too much additional space in an already rather long review paper. A simple definition in lay words is given in the discussion of Fig. 6, ll. 415-416, along with suitable references. A similar effort will be made for Fig. 8.

- ll. 569-619: the authors present here an extensive list of possible applications of the BraMAH approach, but this has not been yet described in the manuscript. I think this part shall be significantly reduced;

We beg to differ. In fact, the comments of the second solicited reviewer, RC2, request us to expand the discussion on future perspectives. We prefer to listen to the advice in RC2.

- l. 671: the authors imply that the method has been adopted and described in Sect. 3.1, but it is not the case (see my point above);

Line 671, which is part of Sec. 3.2, is a bit confusing, since BraMAH for autonomous systems was, in fact, described in the preceding Sec. 3.1 but the templex will be described in the subsequent Sec. 3.3. The sentence will be modified to clarify this point.

- l. 681: it would be nice to see how these classes emerge in the DDG model;

These clouds are obtained by integrating Shadden's ordinary differential equations from different initial conditions. Further details will be provided in the text to save the reader the trouble of having to refer to the source article.

- Figure 13: not clear to me what the colors refer to here, as the label refers to colors that do not appear to be present in the figure;

We will provide this information in the caption or text, to make this review article as self-contained as possible.

- ll. 683-689: are the authors discussing Figure 13 or 14 here;

We are referring to both figures at the same time. We will add color labels below each point cloud in Figure 13 to make the point clear. Thank you for this remark.

- ll. 727-728: to this point, it is not clear to me what a "strip" is in a topological sense. Given that a reader might not be familiar with algebraic topology, I think that some qualitative description might be provided here or elsewhere;

A classical strip in topology is equivalent to a cylinder. We will use the term cylinder instead.

TECHNICAL CORRECTIONS

- l. 141: "they" → "the"; - l. 187: "eingenvalues" → "eigenvalues";

OUTCOMES

Overall, I think that the approach of random temples on random attractors is very promising, and I see potential applications on several aspects that are of interest for climate sciences. I list here a number of possible topics to develop:

- *Design of optimal ensembles for climate predictions: given that in the range of problems related to climate prediction we are not constrained about initial conditions as in NWP, but even if we are in a genuinely non-autonomous dynamical system with a possibly random forcing, we are reasonably confident that the evolution of the attractor will preserve its homologies. This said, an efficient mapping of the ensemble initial conditions on the cloud of trajectories around the fixed point of the attractor, selected according to their homological properties, might help increasing the reliability of ensemble prediction with a reduced number of members. This is well inside the scope of reconciling the different flavors and approaches to Low Frequency Variability, as outlined in the text;*

- *Investigation of precursors of tipping points: given the rigorous definition of "topological tipping points", it would make sense, as outlined in the concluding remarks, to discuss to what extent these tipping points are representative of tipping points in a climatic sense. In order to do so, idealized conceptual models of key tipping elements might be useful tools, as they would bear a relatively known attractor, at the same time allowing to explain the physics behind the described processes and to identify precursors of critical transitions;*

We are grateful to the reviewer for raising these two broader points. Both of them are related to the information provided by random templexes computed for random attractors to the investigation of the actual climate system and of its prediction. We will give the matter some more thought and try to add a few sentences or a couple of paragraphs that might withstand the test of time. Particular attention will be given to the possible connections between TTPs, on the one hand, and the better understood tipping points associated with the description and prediction of the climate system by differential systems rather than by homologies, on the other.