Variational Techniques for a One-Dimensional Energy Balance Model - Supplementary Material

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Introduction

1 Functional definition

The 1D-EBM we consider through the main paper has the form ([2]):

$$\begin{aligned} C_T \partial_t u &= \partial_x \left(\kappa(x) u_x\right) + Q_0(x)\beta(u) + q - \varepsilon_0 \sigma_0 u^4, \quad (t,x) \in [0,T] \times [-1,1] \\ u_x(t,-1) &= u_x(t,1) = 0, \quad t \geq 0, \\ u(0,x) &= \tilde{u}(x), \end{aligned}$$

where $\tilde{u}$ denotes the initial condition. The steady-state solutions to the previous problem are associated with the functional ([3, 6]):

$$F_q(u) = \int_{-1}^{1} \varepsilon_0 \sigma_0 \frac{(u^5)}{5} - Q_0(x) B(u) - qu \, dx + \frac{1}{2} \int_{-1}^{1} \kappa(x) \left[u'(x)\right]^2 \, dx.$$

In the main manuscript, we have illustrated some properties of the minimizer of the variational problem

$$\inf \left\{ F_q(u) \ \mid \ u \in H^1, \ u \geq 0 \right\}.$$

But we can prove more. Indeed, we will extend F_q to a functional defined on H^1, and then shows that there exists $u_0 \in H^1$ s.t. $u_0 \geq 0$ and

$$\tilde{F}_q(u_0) = \inf \left\{ \tilde{F}_q(u) \ \mid \ u \in H^1 \right\} = \inf \left\{ F_q(u) \ \mid \ u \in H^1, \ u \geq 0 \right\}.$$

The necessity to extend F_q to take into account also negative values for u comes from the fact that the natural space in which set the minimization problem is the Sobolev space $H^{1,2}(-1,1)$. But, due to the presence of the odd polynomial term in F_q, we have

$$\inf \left\{ F_q(u) \;\mid\; u \in H^1 \right\} = -\infty.$$

In fact, choosing the constant function $u_\lambda \equiv -\lambda$, with $\lambda > 0$, we get $\lim_{\lambda \to +\infty} F_q(u_\lambda) = -\infty$. This is not surprising, since the term u^5 inside the functional F_q comes from the Stefan-Boltzmann law, which has no sense for negative values of the temperature. Further, in order to pick a primitive B of the co-albedo β, we need to extend the definition of the co-albedo also for the negative values of the Kelvin temperature. In any case, these extensions do not really affect the EBM, since are referred to negative values of u, which have no physical sense.

Let's turn to give details about the extensions of β and the Stefan-Boltzmann law. Let $\tilde{\beta}\colon \mathbb{R} \to \mathbb{R}$ be s.t. $\tilde{\beta} \in C^\infty$ and

(i) $\tilde{\beta}$ is monotonically increasing

(ii) $\tilde{\beta}(u) \geq 0 \; \forall u \in \mathbb{R}$

(iii) $\tilde{\beta}(u) = 0$ for $u \leq -M$, for some $M > 0$

(iv) $\tilde{\beta}$ extends β for $u \geq 0$.

Denote by $B(u) := B_0 + \int_0^u \tilde{\beta}(v)dv$ a primitive of β, with B_0 s.t. $B(u) \geq 0 \; \forall u \in \mathbb{R}$. It will be useful in future to note that

$$0 \leq B(u) \leq |B_0| + |u| \quad \forall u. \tag{1}$$

In a similar way, we consider the extension of the Stefan-Boltzmann law given by:

$$\psi(u) = \begin{cases} \varepsilon_0 \sigma_0 u^4, & \text{if } u \geq 0 \\ 0, & \text{if } u < 0. \end{cases}$$

A primitive is given by $\Psi(u) = \varepsilon_0 \sigma_0 \frac{(u^5)_+}{5}$, where $(x)_+ = max(x,0)$.

Lastly, since in our model we are assuming κ continuous and positive on $[-1,1]$, we can also assume κ constant. Indeed, all the proofs, that in this manuscript are carried with constant κ, extended immediately to the non-constant case. In conclusion, the uniformly elliptic equation we are considering is given by:

$$\begin{aligned} 0 &= \kappa \Delta u + Q_0(x)\tilde{\beta}(u) + q - \psi(u), \\ u'(-1) &= u'(1) = 0, \end{aligned} \tag{2}$$

and the functional associated with its solution is

$$\tilde{F}_q(u) = \int_{-1}^{1} \Psi(u) - Q_0(x)B(u) - qu\,dx + \frac{\kappa}{2}\int_{-1}^{1} \left[u'(x)\right]^2 \,dx.$$

2 Gibbs invariant measure and functional minimum point

In this section, we make rigorous the relation between F_q and the invariant measure ν of the stochastic EBM. In particular, we prove a result giving information about the concentration of ν around minimum points for $\tilde{F}_q$. We start by recalling the notation and some useful results.

First, we set

$$I(u) := \int_{-1}^{1} \left(\varepsilon_0 \sigma_0 \frac{(u^5)_+}{5} - Q_0(x) B(u) - qu \right) dx.$$

Consider $H = L^2(-1,1)$ and $E = C([-1,1])$. Then, following the theory of stochastic partial differential equation (SPDE), the stochastic equation obtained by adding a cylindrical Brownian motion is a gradient SPDE of the form

$$dX_t = [AX_t + f(x, X_t)]\, dt + \varepsilon dW_t, \quad X_{|t=0} = x_0 \tag{3}$$

where $(W_t)_t$ is a cylindrical Wiener process on H and $A = \kappa\Delta$ is the Neumann Laplacian with constant viscosity $\kappa > 0$, i.e. $A\colon D(A) \subset H \to H$,

$$\begin{aligned} D(A) &= \left\{ u \in H^2(-1,1) \ \mid \ u'(-1) = u'(1) = 0 \right\} \\ Au &= \kappa u'', \end{aligned} \tag{4}$$

and

$$f(x,u) = Q_0(x)\tilde{\beta}(u) + q - \varepsilon_0\sigma_0 (u^4)_+$$

We refer to [4] for details about the properties of the previous SPDE. The mild solution X_t of (3) is $\mathbb{P}-$a.s valued in E. Further, applying the theory of invariant measure developed in [7], we get the following property.

Proposition 1 *The SPDE* (3) *has a unique Gibbs invariant measure ν. Further, $\nu \ll \mu$ with explicit formula:*

$$\nu(du) = \frac{1}{Z} \exp\left(-\frac{2}{\varepsilon^2} I(u) \right) \mu(du), \quad u \in H \tag{5}$$

where $\mu \sim \mathcal{N}\left(0, -\frac{\varepsilon^2}{2}A^{-1}\right)$ is a Gaussian measure on H.

Remark 2 *The Neumann Laplacian Δ is not invertible on $H = L^2(-1,1)$ and the invariant measure theory applies for a strictly negative definite operator A on H. For this reason, we should consider the strictly negative operator*

$$\tilde{A} := \lambda Id - A, \quad \lambda > 0.$$

In this way, the functional takes the form:

$$\tilde{I}(u) = I(u) - \lambda \|u\|_2^2,$$

and the reaction term in the SPDE is given by:

$$\tilde{f}(x,u) = f(x,u) - \lambda u.$$

In conclusion, the invariant measure for the SPDE

$$d\tilde{X}_t = \left[\tilde{A}\tilde{X}_t + \tilde{f}(x, X_t)\right] dt + \varepsilon dW_t, \quad \tilde{X}_{|t=0} = \tilde{x}_0$$

is given by:

$$\tilde{\nu}(du) = \frac{1}{Z} \exp\left(-\frac{2}{\varepsilon^2}\tilde{I}(u)\right)\tilde{\mu}(du), \quad u \in H,$$

where $\tilde{\mu} \sim \mathcal{N}(0, -\frac{\varepsilon^2}{2}\tilde{A}^{-1})$ *is a Gaussian measure on* H*. Since this change only complicates the notation in the proofs, we will keep writing* A^{-1} *but the reader should interpret the Laplacian with the shift described above, in order to get the rigorous meaning.*

Second, keeping in mind the previous remark, we adopt from now on the notation:

$$\mathcal{Q} = -\frac{\varepsilon^2}{2\kappa}\Delta^{-1}, \qquad \mu\left(du; v, \mathcal{Q}\right) \sim \mathcal{N}\left(v, \mathcal{Q}\right).$$

The following statement is a classical result about the equivalence of Gaussian measures. See [7] for more details.

Theorem 3 (Cameron-Martin) *The Gaussian measures* $\mu\left(du; 0, \mathcal{Q}\right)$ *and* $\mu\left(du; v, \mathcal{Q}\right)$ *on* H *are equivalent if and only if* $v \in \mathcal{Q}^{1/2}(H)$*. In this case:*

$$\frac{\mu\left(du; 0, \mathcal{Q}\right)}{\mu\left(du; v, \mathcal{Q}\right)} = \exp\left(-\langle \mathcal{Q}^{-1/2}u, \mathcal{Q}^{-1/2}v\rangle + \frac{1}{2}\left\|\mathcal{Q}^{-1/2}v\right\|_2^2\right). \tag{6}$$

In the following, we are going to recall the rigorous meaning for

$$\langle \mathcal{Q}^{-1/2}u, \mathcal{Q}^{-1/2}v\rangle_{L^2}.$$

Consider

$$W_z \colon \mathcal{Q}^{1/2}(H) \subset H \to L^2(H, \mu), \quad W_z(u) := \langle u, \mathcal{Q}^{-1/2}z\rangle_{L^2}.$$

It can be shown that:

(i) W_z is an isometry,

(ii) $Q^{1/2}(H)$ is dense in H (here it is fundamental $\ker(\mathcal{Q}) = \{0\}$.)

In this way, W_z can be extended in a unique way to a map $W_z \colon H \to L^2(H, \mu)$. So, it should be interpreted as:

$$\langle \mathcal{Q}^{-1/2}u, \mathcal{Q}^{-1/2}v\rangle_{L^2} = W_{Q^{-1/2}v}(u), \quad u \in H.$$

Remark 4 *For our choice of the operator* $\mathcal{Q}$*, we have that the Cameron-Martin space is* $\mathcal{Q}^{1/2}(L^2) = H^1$.

At this point, we move to prove the main result of this section. Given a Banach space X, we denote by

$$B_X(x_0, \rho) = \{x \in X \ | \ \|x - x_0\|_X < \rho\}$$

the open ball with center $x_0 \in X$ and radius $\rho > 0$.

Proposition 5 *Let* $C > 0$*,* $r > 5$ *and* $v \in H^{2,2}(-1, 1)$*. Consider the set*

$$B_C(v, \eta) := B_{L^2}(v, \eta) \cap B_{L^r}(v, C).$$

Then,

(i) $\nu(B_C(v, \eta)) \xrightarrow{C \to +\infty} \nu(B_{L^2}(v, \eta))$

(ii) $\mu(B_C(0, \eta)) \xrightarrow{C \to +\infty} \mu(B_{L^2}(0, \eta))$

(iii) For each $C > 0$*, it holds*

$$\frac{\nu(B_C(v, \eta))}{\mu(B_C(0, \eta))} = \frac{1}{Z} exp\left(-\frac{2}{\varepsilon^2}\left(\tilde{F}_q(v) + O(\eta^\theta)\right)\right),$$

where $\theta \in (0, 1)$ *satisfies*

$$\frac{1}{5} = \frac{\theta}{2} + \frac{1 - \theta}{r}.$$

Proof. Assume for simplicity $\kappa = 1$.

(i)-(ii) Observe that for $v_1 \in L^2 \cap L^r$, we have:

$$B_{C_1}(v_1, \eta) \subset B_{C_2}(v_1, \eta), \quad \text{if } C_1 \leq C_2$$

and

$$B_{L^2}(v_1, \eta) = \left(\bigcup_{C>0} B_C(v_1, \eta)\right) \cup \left\{u \in L^2 \ | \ \|u\|_r = \infty\right\}.$$

Denote by $B = \left\{u \in L^2 \ | \ \|u\|_r = \infty\right\}$. If we are able to prove:

$$\mu\,(B) = 0,$$

then we get (i) and (ii) thanks to the continuity of measures on an increasing sequence of sets. Since $\mu \sim \mathcal{N}(0, -\frac{\varepsilon^2}{2}\Delta^{-1})$, then

$$\mu = M \sum_{n=1}^{\infty} \frac{Z_n}{n} e_n$$

where $\{Z_n\}_n$ are i.i.d. $\mathcal{N}(0, 1)$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\{e_n\}_n$ is an orthonormal basis of $H = L^2(-1, 1)$ and $M > 0$ is a constant. The previous series is convergent in $L^2((\Omega, \mathcal{F}, \mathbb{P}); H^s)$, for all $s < 1/2$; indeed

$$\|\mu\|^2_{H^s} = \left\| M \sum_n \frac{Z_n}{n} (-\Delta)^s e_n \right\|_2^2 = \left\| M \sum_n \frac{Z_n}{n} n^s e_n \right\|_2^2 = M^2 \sum_n M \frac{Z_n^2}{n^2} n^{2s} = \sum_n \frac{Z_n^2}{n^{2-2s}}.$$

Taking the expected values, we get

$$\mathbb{E}\|\mu\|^2_{H^s} = M^2 \sum_n \frac{1}{n^{2-2s}} < \infty \;\leftrightarrow\; s < 1/2.$$

The Sobolev embedding $H^s(-1,1) \hookrightarrow L^p$, holds for each $p < \infty$ if s is close to $1/2$. This leads to $\mu(B) = 0$.

(iii) We start by using the explicit formula (5) in order to get

$$\frac{\nu(B_C(v,\eta))}{\mu(B_C(0,\eta))} = \frac{1}{Z} \frac{\int_{B_C(v,\eta)} \exp\left(-\frac{2}{\varepsilon^2} I(u)\right) \mu(du; 0, \mathcal{Q})}{\mu(B_C(0,\eta); 0, \mathcal{Q})}. \tag{7}$$

Using the Cameron-Martin formula (6), we have

$$\mu(du; 0, \mathcal{Q}) = \exp\left(-W_{\mathcal{Q}^{-1/2}v} + \frac{1}{2}\left\|\mathcal{Q}^{-1/2}v\right\|_2^2\right) \mu(du; v, \mathcal{Q})$$

for each $v \in Q^{1/2}(L^2) = H^1$. Since

$$\mathcal{Q}^{-1/2}u = \frac{\sqrt{2}}{\varepsilon} (-\Delta)^{1/2},$$

and for $v \in H^2$

$$W_{\mathcal{Q}^{-1/2}v}(u) = \langle u, \mathcal{Q}^{-1/2}\mathcal{Q}^{-1/2}v\rangle = \frac{2}{\varepsilon^2}\langle u, v''\rangle$$

we arrive to

$$\mu(du; 0, \mathcal{Q}) = \exp\left(-\frac{2}{\varepsilon^2}\langle u', v'\rangle + \frac{1}{\varepsilon^2}\|v'\|_2^2\right) \mu(du; v, \mathcal{Q}).$$

Plugging the previous identity into (7), we deduce:

$$\frac{\nu(B_C(v,\eta))}{\mu(B_C(0,\eta))} = \frac{1}{Z} \frac{\int_{B_C(v,\eta)} \exp\left[-\frac{2}{\varepsilon^2}\left(I(u) + \langle u', v'\rangle - \frac{1}{2}\|v'\|_2^2\right)\right] \mu(du; v, \mathcal{Q})}{\mu(B_C(0,\eta); 0, \mathcal{Q})}$$

Assume for a moment that we are able to prove:

$$-\frac{2}{\varepsilon^2}\left(I(u) + \langle u', v'\rangle - \frac{1}{2}\|v'\|_2^2\right) = -\frac{2}{\varepsilon^2}\left(\tilde{F}_q(v) + O(\eta^\theta)\right), \quad u \in B_C(v,\eta), \tag{8}$$

for $\theta \in (0,1)$. Then,

$$\begin{aligned}\frac{\nu(B_C(v,\eta))}{\mu(B_C(0,\eta))} &= \frac{1}{Z}\exp\left(-\frac{2}{\varepsilon^2}\left(\tilde{F}_q(v) + O(\eta^\theta)\right)\right) \frac{\mu(B_C(v,\eta); v, \mathcal{Q})}{\mu(B_C(0,\eta); 0, \mathcal{Q})} \\ &= \frac{1}{Z}\exp\left(-\frac{2}{\varepsilon^2}\left(\tilde{F}_q(v) + O(\eta^\theta)\right)\right),\end{aligned}$$

where we have used

$$\mu(B_C(v,\eta); v, \mathcal{Q}) = \mu(B_C(0,\eta); 0, \mathcal{Q}).$$

This concludes the proof. ■

It remains to prove (8).

Lemma 6 *If $u \in B_C(v,\eta)$ and $v \in H^2$ then there exists $\theta \in (0,1)$ s.t.*

$$-\frac{2}{\varepsilon^2}\left(I(u) + W_{\mathcal{Q}^{-1/2}v}(u) - \frac{1}{2}\|v'\|_2^2\right) = -\frac{2}{\varepsilon^2}\left(\tilde{F}_q(v) + O(\eta^\theta)\right).$$

Proof. Assume for simplicity $\kappa = 1$. We divide the proof into steps.

Step 1: $W_{\mathcal{Q}^{-1/2}v}(u) = \langle v', v'\rangle + O(\eta)$, if $u \in B_C(v,\eta)$ and $v \in H^2$.

Indeed, since $v \in H^2$, we have $W_{\mathcal{Q}^{-1/2}v}(u) = -\langle u, v''\rangle$ and

$$|-\langle u, v''\rangle - \langle v', v'\rangle| = |-\langle u, v''\rangle + \langle v, v''\rangle| = |\langle v-u, v''\rangle| \le \|v-u\|_2\|v\|_{H^2} \le \eta\|v\|_{H^2}.$$

Step 2 : $I(u) = I(v) + O(\eta^\theta)$, if $u \in B_C(v,\eta)$.

Observe that:

$$|I(u) - I(v)| \le \frac{\varepsilon_0\sigma_0}{5}\int_{-1}^{1}\left|(u^5)_+ - (v^5)_+\right| dx + \int_{-1}^{1} Q_0(x)|B(u)-B(v)|dx + q\int_{-1}^{1}|u-v|dx$$

By the properties of B and Q_0, we get that there exists $M, M' > 0$ s.t.

$$\int_{-1}^{1} Q_0(x)|B(u)-B(v)|dx + q\int_{-1}^{1}|u-v|dx \le M\int_{-1}^{1}|u-v|\,dx \le M'\|u-v\|_2 \le M'\eta.$$

By the mean value theorem, we get that if $u, v \ge 0$ and $p \ge 1$, then

$$|u^p - v^p| \le p\max\{|u|,|v|\}^{p-1}|u-v| \le p\,(|u|+|v|)^{p-1}|u-v|.$$

By this inequality, we get

$$\int_{-1}^{1}|(u^p)_+ - (v^p)_+|\,dx \le p\int_{-1}^{1}(u_+ + v_+)^{p-1}|u_+ - v_+|\,dx.$$

Let q_1 s.t. $\frac{1}{p} + \frac{1}{q_1} = 1$. By Holder's inequality, we deduce:

$$\begin{aligned}\int_{-1}^{1}(u_+ + v_+)^{p-1}|(u)_+ - (v)_+|\,dx &\le \left[\int_{-1}^{1}(u_+ + v_+)^{q_1(p-1)}\,dx\right]^{1/q_1}\|u_+ - v_+\|_p \\ &\le \|u_+ + v_+\|_p^{p/q_1}\|u_+ - v_+\|_p \\ &\le \left(\|u\|_p + \|v\|_p\right)^{p/q_1}\|u-v\|_p.\end{aligned}$$

Choosing $p = 5$, by interpolation inequality there exists $\theta \in (0,1)$ s.t.

$$\|u-v\|_p \le \|u-v\|_2^\theta\|u-v\|_r^{1-\theta} \le \eta^\theta C^{1-\theta}, \quad u \in B_C(v,\eta).$$

In this way, for $u \in B_C(v,\eta)$, we deduce

$$\int_{-1}^{1}\left|u_+^5 - v_+^5\right|dx \le p\left(\|u\|_p + \|v\|_p\right)^{p/q_1}\|u-v\|_p \le p\left(C + 2\|v\|_p\right)^{p/q_1}\eta^\theta C^{1-\theta} = O(\eta^\theta).$$

■

3 Variational problem - existence

Given a Banach space X and a sequence $\{u_n\}_n \subseteq X$, we denote by $u_n \rightharpoonup u$ the weak convergence, while we reserve the symbol $u_n \to u$ for strong convergence. Further, $H^1 = H^{1,2}(-1,1)$ will denote the Sobolev Space on $[-1,1]$ with order 1 and exponent 2. The main result of this section is the following.

Proposition 7 *Assume $q > 0$. Then, the variational problem*

$$\inf\left\{\tilde{F}_q(u) \ \mid \ u \in H^{1,2}\right\} \tag{9}$$

admits a minimizer u_0. Further, $u_0 \in C^\infty$, $u_0'(-1) = u_0'(1) = 0$ and $u_0 \geq 0$.

Proof. Let assume for simplicity $Q_0(x) = 1\ \forall x$. This is not restrictive and the proof can be carried on in a similar way since

$$Q_0(x) > \delta > 0.$$

We divide the proof into steps.

Step 1: compactness. We consider the notion of convergence on $\mathbb{X}$ given by:

$$u_n \xrightarrow{\mathbb{X}} u_\infty \quad \text{if and only if } u_n \to u_\infty \text{ uniformly in } [-1,1] \text{ and } u_n' \rightharpoonup u_\infty' \ \text{ in } L^2.$$

We want to verify the compactness of the sublevel sets of $\tilde{F}_q$. Let $\{u_n\}_n \subset \mathbb{X}$ and $M > 0$ s.t $M \geq \tilde{F}_q(u_n)\ \forall n$. First, we observe

$$\begin{aligned} M \geq \tilde{F}_q(u_n) &\geq \int_{-1}^{1} \varepsilon_0\sigma_0 \frac{(u_n^5)_+}{5} - B(u_n) - qu_n\, dx \overset{\text{Lagrange Thm}}{=} 2\left[\varepsilon_0\sigma_0\frac{(u_n^5)_+}{5}(\xi_n) - B(u_n)(\xi_n) - qu_n(\xi_n)\right] \\ &\geq 2\left[\varepsilon_0\sigma_0\frac{(u_n^5)_+}{5}(\xi_n) - B_0 - |u_n(\xi_n)| - qu_n(\xi_n)\right], \end{aligned}$$

where $\xi_n \in [-1,1]$. Since $v \mapsto \varepsilon_0\sigma_0(v^5)_+/5 - B(v) - qv$ explodes for $v \to \pm\infty$, we get the existence of $C_1 > 0$ s.t. $|u_n(\xi_n)| \leq C_1\ \forall n$. Second, we get $\forall x \in [-1,1]$

$$|u_n(x)| \leq |u_n(\xi_n)| + |u_n(x) - u_n(\xi_n)| \leq C_1 + \|u_n'\|_2 |x - \xi_n|^{1/2}, \tag{10}$$

the second inequality follows from the fact that a function in H^1 is Holder-continuous. Third, since (1) holds, we have

$$M \geq \tilde{F}_q(u_n) \geq \int_{-1}^{1} -B(u_n) - u_n\, dx + \frac{1}{2}\|u_n'\|_2 \gtrsim -\|u_n\|_1 + \frac{1}{2}\|u_n'\|_2 \gtrsim -C_1 - \|u_n'\|_2 + \frac{1}{2}\|u_n'\|_2^2,$$

where $a \gtrsim b$ if and only if exists $c > 0$ s.t. $a \geq c \cdot b$ and the last inequality follows from $\|u_n\|_1 \lesssim \|u_n\|_2$ .The previous inequality of second order in the unknown $\|u_n'\|_2$ is verified if and only if

$$\|u_n'\|_2 \leq C_2, \tag{11}$$

for some $C_2 > 0$. Up to remaining the subsequence, we have $u'_n \rightharpoonup v$, for a $v \in H^1$. It remains to prove the uniform converge of u_n in $[-1, 1]$. Let's do it using the Ascoli-Arzelà theorem. We get equi-continuity from the properties of the Sobolev space. Indeed

$$|u_n(x) - u_n(y)| \leq \|u'_n\|_{L^2}|x - y|^{1/2} \quad \forall x, y \in [-1, 1]$$

and $\|u'_n\|_{L^2}$ is bounded thanks to weak convergence. Since (10) holds, we get also equi-boundedness. Then (up to remaining) $u_n \to u_\infty$ uniformly in $[-1, 1]$.

It remains to prove $u'_\infty = v$ in weak sense. Let $\phi \in C_c^\infty([-1, 1])$. Then, by weak derivative definition,

$$\int_{-1}^{1} u_n \phi' \, dx = -\int_{-1}^{1} u'_n \phi \, dx \quad \forall n,$$

and taking the limit on both sides of the equality (we use uniform convergence at LHS, and weak convergence at RHS)

$$\int_{-1}^{1} u_\infty \phi' \, dx = -\int v \phi \, dx$$

Step 2: lower semi-continuity of $\tilde{F}_q$. Let $\{u_n\} \subset \mathbb{X}$ be s.t. $u_n \xrightarrow{\mathbb{X}} u$. Let F_1, F_2 be s.t.

$$\tilde{F}_q(u) = F_1(u) + F_2(u), \quad F_1(u) := \int_{-1}^{1} \frac{\varepsilon_0 \sigma_0 (u^5)_+}{5} - B(u) - qu \, dx, \quad F_2(u) := \frac{\kappa}{2} \int_{-1}^{1} (u')^2 \, dx$$

By uniform convergence, we have $\lim_{n\to\infty} F_1(u_n) = F_1(u)$; by lower semi-continuity of the L^2 norm w.r.t. weak convergence, we have $\liminf_{n\to\infty} F_2(u_n) \geq F_2(u)$.

In conclusion, $\tilde{F}_q$ is lower semi-continuous and coercive. Then, $\exists\, u_0 \in \mathbb{X}$ minimum point for $\tilde{F}_q$ in $\mathbb{X}$.

Step 3: regularity for u_0 The first variation of $\tilde{F}_q$ in the point u in direction h is given by:

$$\delta \tilde{F}_q(u, h) = \int_{-1}^{1} \Big(\psi(u) - \tilde{\beta}(u) - q \Big) h \, dx + \kappa \int_{-1}^{1} u' h' \, dx.$$

Choosing $h \in C_c^\infty([-1, 1])$ and setting $\phi(t) := F_q(u_0 + th)$, it holds $\phi'(0) = 0$. So:

$$0 = \phi'(0) = \delta \tilde{F}_q(u_0, h),$$

from which it follows

$$\int_{-1}^{1} \Big(\psi(u_0) - \tilde{\beta}(u_0) - q \Big) h \, dx = -\kappa \int_{-1}^{1} u'_0 h' \, dx. \tag{12}$$

Then

$$\kappa u''_0 = \psi(u_0) - \tilde{\beta}(u_0) - q \tag{13}$$

in the weak sense. The RHS is C^0 because $u_0 \in H^1$. Then $u_0' \in C^1$ and $u_0 \in C^2$. Repeating the bootstrap argument, we get $u_0 \in C^\infty$.

Step 4: Neumann boundary conditions. Let $h \in C^\infty$. Following the same arguments above, we get to (12). Integrating by parts the RHS, we have:

$$\int_{-1}^{1} \left(\psi(u_0) - \tilde{\beta}(u_0) - q - u_0''\right) h\, dx = -\kappa \left(h(1)u_0'(1) - h(-1)u_0'(-1)\right)$$

But the LHS of the previous equation is null thanks to (13). Choosing h s.t. $h(1) = 0$ and $h(-1) \neq 0$, it follows $u_0'(-1) = 0$. In a similar way, we can get $u_0'(1) = 0$.

Step 5: $u_0 \geq 0$. This can be proved by the following truncation argument. Assume there exists $x_0 \in [-1, 1]$ s.t. $u_0(x_0) < 0$. Consider the following points

$$\tau_1 := \sup\{x < x_0 \ \mid u_0(x) = 0\}, \quad \tau_2 := \inf\{x > x_0 \ \mid u_0(x) = 0\}.$$

Let $\tilde{u}_0$ be the truncation to 0 of u_0 in $[\tau_1, \tau_2]$, i.e.

$$\tilde{u}_0(x) := \begin{cases} u_0(x) & \text{if } x \in [\tau_1, \tau_2]^c, \\ 0 & \text{if } x \in [\tau_1, \tau_2]. \end{cases} \tag{14}$$

Then

$$\begin{aligned} \tilde{F}_q(u_0) - \tilde{F}_q(\tilde{u}_0) &= \int_{\tau_1}^{\tau_2} \varepsilon_0\sigma_0 \frac{(u_0^5)_+}{5} - Q_0(x)B(u_0) - qu_0(x)\, dx + \frac{\kappa}{2}\int_{\tau_1}^{\tau_2} [u_0']^2\, dx + \int_{\tau_1}^{\tau_2} Q_0(x)B(0)\, dx \\ &\geq \int_{\tau_1}^{\tau_2} Q_0(x)\left(B(0) - B(u_0)\right) - qu_0(x)\, dx > 0 \end{aligned}$$

where the last inequality follows from the fact that $B(0) \geq -B(u_0(x))$ and $q > 0$. This is a contradiction and thus $u_0(x) \geq 0$ $\forall x \in [-1, 1]$. ■

Repeating again a truncation argument similar to the one used in the last part of the previous proof, we can prove that the minimizer is bounded from above.

Lemma 8 *Assume $q \in (0, b)$. Then, there exists $M > 0$ s.t. if $u_0 = u_0(q)$ is the minimizer for $\tilde{F}_q$, then $u_0 \leq M$.*

Proof. Indeed, set $\mathcal{R}(x, u) := \varepsilon_0\sigma_0 \frac{(u^5)_+}{5} - Q_0(x)B(u) - qu$. Note that the following inequalities hold

$$\begin{aligned} \mathcal{R}(x, u) &\geq \varepsilon_0\sigma_0 \frac{u^5}{5} - \|Q_0\|_\infty B(u) - bu, \quad u \geq 0, \\ \mathcal{R}(x, v) &\leq \varepsilon_0\sigma_0 \frac{v^5}{5}, \quad v \geq 0. \end{aligned}$$

So, if we set $G(x, u, v) := \mathcal{R}(x, u) - \mathcal{R}(x, v)$, we have, uniformly in x

$$G(x, u, v) \geq \varepsilon_0\sigma_0 \left(\frac{u^5}{5} - \frac{v^5}{5}\right) - \|Q_0\|_\infty B(u) - bu.$$

Note that for each $v \geq 0$ fixed the term on the right-hand side diverges to $+\infty$ for $u \to +\infty$. So, given $v \geq 0$, there exists $M = M(v)$ s.t if $u \geq M(v)$, then $G(x, u, v) \geq 1 \; \forall \, x \in [-1, 1]$.

Let us pick $v = 1$. We want to prove that $u_0(x) \leq M \; \forall x \in [-1, 1]$. Assume by contradiction that $u_0(x_0) > M$ for some $x_0 \in [-1, 1]$. Set

$$\tau_1 := \sup \{x < x_0 \; \mid u_0(x) = M\}, \quad \tau_2 := \inf \{x > x_0 \; \mid u_0(x) = M\},$$

and consider the truncated minimizer

$$\tilde{u}_0(x) := \begin{cases} u_0(x) & \text{if } x \in [\tau_1, \tau_2]^c, \\ M & \text{if } x \in [\tau_1, \tau_2]. \end{cases} \tag{15}$$

Then

$$\tilde{F}_q(u_0) - \tilde{F}_q(\tilde{u}_0) \geq \int_{\tau_1}^{\tau_2} \mathcal{R}(x, u_*(x)) - \mathcal{R}(x, M) dx = \int_{\tau_1}^{\tau_2} G(x, u_*(x), M) dx.$$

Since by construction $u_0(x) \geq M \; \forall x \in [\tau_1, \tau_2]$, then we conclude

$$\tilde{F}_q(u_0) - \tilde{F}_q(\tilde{u}_0) > 0.$$

This concludes the proof. ■

4 Variational problem - uniqueness

In this section we are going to characterize the uniqueness for the solution of the variational problem in terms of the value function, i.e. the minimum value attained by $\tilde{F}_q$ on H^1.

The *value function* V is defined as follows:

$$V(q) = \inf \left\{ \tilde{F}_q(u) \colon u \in H^1 \right\}.$$

First of all, from the last result in Section 3 follows the Lipschitz property for V.

Lemma 9 *Assume $q \in (0, b)$. Then, the value function $q \mapsto V(q)$ is Lipschitz continuous.*

Proof. First of all, observe that thanks to the non-negativity of the minimizer and Lemma 8 there exists $M > 0$ s.t.

$$V(q) = \inf \{F_q(u) \; \mid \; 0 \leq u \leq M\}.$$

Second, given a family $\{f_i\}_{i \in I}$ of L_i-Lipschitz function f_i, we know that the infimum $\inf_{i \in I} f_i$ is Lipschitz as long as we can bound uniformly the constants L_i. In our case this is true. Indeed, given $u \in H^1$, $0 \leq u \leq M$, we have:

$$\left| \tilde{F}_{q_1}(u) - \tilde{F}_{q_2}(u) \right| = |q_2 - q_1| \int_1^1 |u(x)| dx \leq 2M |q_2 - q_1|.$$

This concludes the proof. ■

The main result of this section is the following. We immediately give its proof and postpone to the remaining part of the section the proof of auxiliary results.

Proposition 10 *Assume $q \in (0, b)$. Then, V is differentiable in μ if and only if there exists an unique minimizer for $\tilde{F}_\mu$ in H^1. Further, if V is differentiable, then*

$$V'(\mu) = -\int_{-1}^{1} u_* dx,$$

with $u_* \in argmin\left\{F_\mu(u) \colon u \in H^1\right\}$.

Proof. $\Rightarrow$) Let's consider the auxiliary function

$$W : \mathbb{R} \times (H^1 \cap \{u \geq 0\}) \to \mathbb{R}$$

given by:

$$W(q, u) := \tilde{F}_q(u) - V(q).$$

As preliminary remarks, note that $W(q, u) \geq 0$ and, if $\tilde{u} \in argmin\left\{\tilde{F}_q(u) \colon u \in H^1\right\}$, then $W(q, \tilde{u}) = 0$. The existence of minimizer has been proved in Proposition 7. To prove uniqueness, assume that u_1, u_2 are two minimizers. Since (i) $\tilde{F}_q$ is differentiable for each q, (ii) V is differentiable in μ by hypothesis and (iii) $W(\mu, u_i) = 0$, $i = 1, 2$, then

$$0 = \partial_q W(\mu, u_i) = -\int_{-1}^{1} u_i(x) dx - V'(\mu).$$

Thus

$$-\int_{-1}^{1} u_1(x) dx = -\int_{-1}^{1} u_2(x) dx = V'(\mu).$$

Using Lemma 10, we know that also $u_1 \wedge u_2$ is a minimizer. With the same reasoning above, it holds

$$-\int_{-1}^{1} u_1(x) dx = -\int_{-1}^{1} u_2(x) dx = -\int_{-1}^{1} u_1 \wedge u_2(x) dx$$

Since $u_i \geq u_1 \wedge u_2 \geq 0$, the previous identities can hold only if $u_1 = u_2$.

$\Leftarrow$) Assume that, given μ, $\exists!$ u_* minimizer for $\tilde{F}_\mu$. Let $\{h_n\}_n$ be a sequence s.t. $h_n \to 0$. Let's denote by u_q a minimizer for $\tilde{F}_q$, i.e.

$$u_q \in argmin\{\tilde{F}_q(u) \colon u \in H^1\},$$

Then, set $u_n := u_{\mu+h_n}$. We are going to show that $V'(\mu) = -\int_{-1}^{1} u_* dx$, i.e.

$$\lim_{n\to\infty} \frac{V(\mu + h_n) - V(\mu)}{h_n} = -\int_{-1}^{1} u_* dx.$$

First, observe that by definition of the value function, we have

$$\frac{V(\mu+h_n)-V(\mu)}{h_n} = \frac{\tilde{F}_{\mu+h_n}(u_n)-\tilde{F}_\mu(u_*)}{h_n} \le \frac{\tilde{F}_{\mu+h_n}(u_*)-\tilde{F}_\mu(u_*)}{h_n} = -\int_{-1}^{1} u_* dx.$$

Hence

$$\lim_{n\to\infty} \frac{V(\mu+h_n)-V(\mu)}{h_n} \le -\int_{-1}^{1} u_* dx.$$

On the other hand,

$$\begin{aligned}\frac{V(\mu+h_n)-V(\mu)}{h_n} &= \frac{\tilde{F}_{\mu+h_n}(u_n)-\tilde{F}_\mu(u_*)}{h_n} \ge \frac{\tilde{F}_{\mu+h_n}(u_n)-\tilde{F}_\mu(u_n)}{h_n} \\ &= -\int_{-1}^{1} u_n dx.\end{aligned}$$

It follows that:

$$\frac{V(\mu+h_n)-V(\mu)}{h_n} \ge -\int_{-1}^{1} u_* dx + \int_{-1}^{1} (u_* - u_n)\, dx$$

But the second integral on the right-hand side converges to zero as $n \to +\infty$ thanks to Lemma 12. This concludes the proof. ■

In order to complete the proof of the previous result, we need to verify some auxiliaries lemmas. First, let's prove that the infimum of two minimizers for $\tilde{F}_q$ is still a minimizer.

Lemma 11 *If u_1, u_2 are minimizers for $\tilde{F}_q$ in H^1, then also $u_1 \vee u_2$ and $u_1 \wedge u_2$ are minimizers.*

Proof. For simplicity of notation, set $\mathcal{R}(x,u) := \varepsilon_0\sigma_0\frac{u^5}{5} - Q_0(x)\beta(u) - qu$. Further, we divide the proof into steps.

Step 1: $\tilde{F}_q(u_1 \wedge u_2) \ge \tilde{F}_q(u_1 \vee u_2)$.

We start observing that:

$$\begin{aligned}\tilde{F}_q(u_1\wedge u_2) &= \frac{\kappa}{2}\int_{u_1\ge u_2}(u_2')^2dx + \int_{u_1\ge u_2}\mathcal{R}(x,u_2)dx + \frac{\kappa}{2}\int_{u_1<u_2}(u_1')^2dx + \int_{u_1<u_2}\mathcal{R}(x,u_1)dx \\ \ge \tilde{F}_q(u_1) &= \frac{\kappa}{2}\int_{u_1\ge u_2}(u_1')^2dx + \int_{u_1\ge u_2}\mathcal{R}(x,u_1)dx + \frac{\kappa}{2}\int_{u_1<u_2}(u_1')^2dx + \int_{u_1<u_2}\mathcal{R}(x,u_1)dx\end{aligned}$$

where the inequality holds since u_1 is a minimizer. So, we deduce

$$\frac{\kappa}{2}\int_{u_1\ge u_2}(u_2')^2dx + \int_{u_1\ge u_2}\mathcal{R}(x,u_2)dx \ge \frac{\kappa}{2}\int_{u_1\ge u_2}(u_1')^2dx + \int_{u_1\ge u_2}\mathcal{R}(x,u_1)dx. \tag{16}$$

In a similar way, we get

$$\frac{\kappa}{2}\int_{u_1<u_2}(u_1')^2dx + \int_{u_1<u_2}\mathcal{R}(x,u_1)dx \ge \frac{\kappa}{2}\int_{u_1<u_2}(u_2')^2dx + \int_{u_1<u_2}\mathcal{R}(x,u_2)dx. \tag{17}$$

Indeed, the previous inequality follows bounding from below $\tilde{F}_q(u_1 \wedge u_2)$ with $\tilde{F}_q(u_2)$ and comparing the terms on both sides of the inequality. Now, adding together equation (16) and equation (17), we obtain the claimed relation between $\tilde{F}_q(u_1 \wedge u_2)$ and $\tilde{F}_q(u_1 \vee u_2)$:

$$\begin{aligned}\tilde{F}_q(u_1 \wedge u_2) &= \frac{\kappa}{2}\int_{u_1 \geq u_2} (u_2')^2 dx + \int_{u_1 \geq u_2} \mathcal{R}(x, u_2) dx + \frac{\kappa}{2}\int_{u_1 < u_2} (u_1')^2 dx + \int_{u_1 < u_2} \mathcal{R}(x, u_1) dx \\ &\geq \frac{\kappa}{2}\int_{u_1 \geq u_2} (u_1')^2 dx + \int_{u_1 \geq u_2} \mathcal{R}(x, u_1) dx + \frac{\kappa}{2}\int_{u_1 < u_2} (u_2')^2 dx + \int_{u_1 < u_2} \mathcal{R}(x, u_2) dx \\ &= \tilde{F}_q(u_1 \vee u_2).\end{aligned}$$

Step 2: $\tilde{F}_q(u_1 \vee u_2) \geq \tilde{F}_q(u_1 \wedge u_2)$.

This inequality can be obtained by repeating the step above starting with $\tilde{F}_q(u_1 \vee u_2)$ instead of $\tilde{F}_q(u_1 \wedge u_2)$.

Step 3: $\tilde{F}_q(u_1) = \tilde{F}_q(u_1 \wedge u_2) = \tilde{F}_q(u_1 \vee u_2)$.

The last identity follows from Step 1 and Step 2. To get the first identity, let's observe that in our case

$$\tilde{F}_q(u_1) + \tilde{F}_q(u_2) = \tilde{F}_q(u_1 \wedge u_2) + \tilde{F}_q(u_1 \vee u_2). \tag{18}$$

The previous identity can be verified by writing

$$\tilde{F}_q(v) = \frac{\kappa}{2}\int_{u_1 \geq u_2} (v')^2 dx + \frac{\kappa}{2}\int_{u_1 < u_2} (v')^2 dx + \int_{u_1 \geq u_2} \mathcal{R}(x, v) dx + \int_{u_1 < u_2} \mathcal{R}(x, v) dx,$$

for $v = u_1, u_2, u_1 \wedge u_2, u_1 \vee u_2$. Then, it just consists in checking that equation (18) holds. At this point, since $\tilde{F}_q(u_1) = \tilde{F}_q(u_2)$ (because u_1, u_2 are minimizers) and $\tilde{F}_q(u_1 \wedge u_2) = \tilde{F}_q(u_1 \vee v_2)$ (thanks to Step 1 and Step 2), equation (18) can be rewritten as:

$$2\tilde{F}_q(u_1) = 2\tilde{F}_q(u_1 \wedge u_2).$$

■

Second, we need to verify that the space integral of a sequence of minimizers behaves in a continuous way as the parameter q approaches a point where the uniqueness hold for the variational problem.

Lemma 12 *Assume $q \in (0, b)$ and that there exists an unique minimizer u_* for $\tilde{F}_\mu$ in H^1. Consider a sequence q_n s.t. $q_n \to \mu$. Then,*

$$\int_{-1}^{1} u_n dx \to \int_{-1}^{1} u_* dx,$$

with $u_n \in argmin \left\{ \tilde{F}_{q_n} : u \in H^1 \right\}$.

Proof. We divide the proof into several steps. Some of them will involve repeating part of the direct method used to solve the variational problem considered in Proposition 7.

Step 1*: there exists* $u_\infty \in H^1$ *and a subsequence* $(n_k)_k$ *s.t.* $u_{n_k} \to u_\infty$ *uniformly in* $[-1,1]$ *and* $u'_{n_k} \rightharpoonup u'_\infty$ *weakly in* L_2*.*

Indeed, since V is continuous thanks to Lemma 9, we have:

$$\tilde{F}_{q_n}(u_n) = V(q_n) \to V(\mu) = \tilde{F}_\mu(u_*).$$

In this way, we infer the existence of $M > 0$ s.t. $\tilde{F}_{q_n}(u_n) \leq M \ \forall n$. At this point, we are in the same hypothesis of the proof for Proposition 7 - Step 1. Following that reasoning, we get the claimed statement.

Step 2*:* $\tilde{F}_\mu(u_n) \to \tilde{F}_\mu(u_*)$*.*

Since u_* is a minimizer for $\tilde{F}_\mu$, we have:

$$\tilde{F}_\mu(u_n) \geq \tilde{F}_\mu(u_*) \quad \forall n.$$

To get the thesis, we fix $\varepsilon > 0$ and we will verify that for n large enough it holds

$$\tilde{F}_\mu(u_n) \leq \tilde{F}_\mu(u_*) + \varepsilon.$$

Indeed

$$\begin{aligned}\left|\tilde{F}_\mu(u_n) - \tilde{F}_\mu(u_*)\right| &\leq \left|\tilde{F}_\mu(u_n) - \tilde{F}_{q_n}(u_n)\right| + \left|\tilde{F}_{q_n}(u_n) - \tilde{F}_\mu(u_*)\right| \\ &\leq |\mu - q_n| \|u_n\|_1 + |V(q_n) - V(\mu)|.\end{aligned}$$

Thanks to Lemma 8, the term $\|u_n\|_1$ is bounded uniformly in n. Further, by the continuity of V we conclude that the right-hand side converges to 0 for $n \to \infty$.

Step 3*:* $u_\infty = u_*$*.*

By proof of Proposition 7 - Step 2, we know that if $u_{n_k} \to u_\infty$ uniformly in $[-1,1]$ and $u'_{n_k} \rightharpoonup u'_\infty$ weakly in L^2, then

$$\liminf_k \tilde{F}_\mu(u_{n_k}) \geq \tilde{F}_\mu(u_\infty).$$

Further, since the sequence $\tilde{F}_\mu(u_n)$ is convergent to $\tilde{F}_\mu(u_*)$, we get

$$\tilde{F}_\mu(u_*) = \lim_n \tilde{F}_\mu(u_n) = \liminf_k \tilde{F}_\mu(u_{n_k}) \geq \tilde{F}_\mu(u_\infty).$$

Thus, by the uniqueness of the minimizer for $\tilde{F}_\mu$, we conclude $u_* = u_\infty$.

Step 4*:* $u_n \to u_*$ *unif. in* $[-1,1]$*. In particular,* $\lim_{n\to\infty} \int_{-1}^1 u_n dx = \int_{-1}^1 u_* dx$*.*

Take a subsequence u_{n_k} of u_n. We can use the same reasoning of Step 1 and get that there exists a subsubsequence $u_{n_{k_h}}$ s.t. $u_{n_{k_h}} \to u_\infty$ uniformly in $[-1,1]$. But with the same reasoning in Step 3, it follows $u_\infty = u_*$. Since the limit does not depend on n_{k_h}, we get the claimed statement. ∎

5 Mountain Pass Theorem and existence of at least three steady-state solutions

In this section, we are going to use the Mountain pass theorem (MPT) from the calculus of variation to show the existence of at least three solutions to the elliptic problem (2). First, we start by checking that the functional $\tilde{F}_q$ satisfies the compactness condition (Palais-Smale) needed in the hypothesis of the MPT. Second, we are going to show how numerical simulations suggest the existence of two (local) minimum points for $\tilde{F}_q$ corresponding to u_S and u_W; thus, the MPT gives us the existence of a third critical point, that corresponds to u_M thanks to numerical simulations. Third, we are giving sufficient hypotheses in order to prove the existence of the two local minimum points mentioned before; the existence of these two local minimum points is again obtained using the direct method.

Let $(X, \|\cdot\|)$ be a reflexive Banach space, $\Phi \in C^1(X, \mathbb{R})$ be a functional and Φ' denote the first variation of Φ.

Definition 13 *The functional Φ satisfies the* Palais-Smale condition *((PS)-condition) if any sequence $\{u_n\}_n \subset X$ s.t.*

$$\Phi(u_n) \text{ is bounded and } \Phi'(u_n) \to 0,$$

admits a convergent subsequence.

The previous one is a compactness condition needed in order to use the Mountain pass theorem, that loosely speaking affirms the existence of a mountain pass between two valleys. See [5] for more details.

Theorem 14 (Mountain pass) *If Φ satisfies the (PS)-condition, $\Phi(0) = 0$ and*

$$\exists \rho, \alpha > 0 \text{ s.t. } \Phi(x) \geq \alpha \quad \forall x \text{ with } \|x\| = \rho,$$

$$\exists\, x_1 \text{ s.t. } \|x_1\| > \rho \text{ and } \Phi(x_1) \leq 0$$

then, $\exists\, x_2$ s.t. $\Phi(x_2) = c \geq \alpha$ and x_2 is a stationary point for Φ. Further,

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} \Phi(u)$$

where:

$$\Gamma = \{\gamma \in C([0,1], X) : \ \gamma(0) = 0, \ \gamma(1) = x_2\}.$$

At this point, we already know from Proposition 7 the existence of a global minimizer for $\tilde{F}_q$. On the other hand, numerical simulations suggest the existence of a second local minimizer. Indeed, the second variation $\delta^2 \tilde{F}_q$ of the functional $\tilde{F}_q$ in the point u in direction h is given by:

$$\delta^2 \tilde{F}_q(u, h) = \int_{-1}^{1} \left[4u^3 h - Q_0(x)\beta'(u)h - \kappa h''\right] h \, dx.$$

We denote by:

$$\lambda_1(u) \leq \lambda_2(u) \leq \cdots,$$

the eigenvalues of the second variation

$$h \mapsto 4u^3 h - Q_0(x)\beta'(u)h - \partial_x \left(\kappa(x) h'\right).$$

We numerically evaluate the eigenvalues of the second variations in the three steady-state points $u_S \leq u_M \leq u_W$. The results, which are shown in Figure 1, tell us that u_S and u_W are strict local minimum points, except at the bifurcations points. This is because the smallest eigenvalues λ_1 for u_S and u_W is positive, hence the second variation in u_S and u_W is positive definite. From this, we get numerical evidence for the existence of a second minimizer. At this point, the Mountain pass theorem guarantees the existence of a third steady-state point, that from our numerical simulations corresponds to u_M, if we are able to prove the (PS)-property for $\tilde{F}_q$. This is what we are going to check in the following.

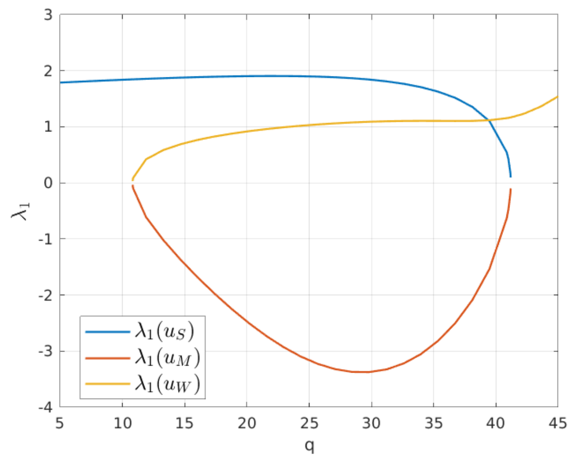

Figure 1: Smallest eigenvalues λ_1 for the second variation in u_S, u_M, u_W.

Let X be a reflexive Banach space and X^* its dual space. Given $x_n, x \in X$, denote by $x_n \rightharpoonup x$ the weak convergence in X.

Definition 15 *$A\colon X \to X^*$ is of* type $(S)_+$ *if any $\{x_n\}_n \subset X$ s.t. $x_n \rightharpoonup x$ and $\limsup_{n\to+\infty} \langle A(x_n), x_n - x\rangle \leq 0$ imply $x_n \to x$ in X.*

Let $X = W_n^{1,2} = W_n^{1,2}(-1,1)$, i.e. the Banach space

$$W_n^{1,2}(-1,1) = \left\{ u \in W^{1,2}(-1,1) \,:\, u = \lim_{n\to\infty} u_n \text{ in } W^{1,2},\ u_n \in C^\infty([-1,1]),\ u_n'(-1) = u_n'(1) = 0 \right\}.$$

Further, let $A\colon X \to X^*$ be the operator given by:

$$\langle A(u), v\rangle = \int_{-1}^{1} u'v'\,dx,$$

where $\langle\cdot,\cdot\rangle = {}_{X^*}\langle\cdot,\cdot\rangle_X$ denotes the duality pairing. We need to recall the following property of the operator A.

Proposition 16 ([1]) *The operator A is of type $(S)_+$.*

At this point, we are able to check the Palais-Smale condition for the functional $\tilde{F}_q$.

Proposition 17 *The functional $\tilde{F}_q\colon W_n^{1,2} \to \mathbb{R}$ satisfies the (PS)-condition.*

Proof. Consider $\{u_n\}_n \subseteq W_n^{1,2}$ and assume there exist $M > 0$ and a sequence $\{\varepsilon_n\}_n$ s.t.

$$\left|\tilde{F}_q(u_n)\right| \leq M, \quad \left\|\tilde{F}_q'(u_n)\right\|_{\left(W_n^{1,2}\right)^*} \leq \varepsilon_n, \tag{19}$$

where $\varepsilon_n > 0$, $\varepsilon_n \to 0$ and

$$\langle \tilde{F}_q'(u), v\rangle = \int_{-1}^{1} \Big(\psi(u)\varepsilon_0\sigma_0 - q - \tilde{\beta}(u)\Big)\, v\, dx - \langle A(u), v\rangle := \int_{-1}^{1} f(x,u)v\, dx - \langle A(u), v\rangle.$$

The proof is divided in two steps.

Step 1: u_n is bounded in $W_n^{1,2}$.

This first step is a corollary of the proof of Proposition 7. Indeed, in that proof, we assume $\tilde{F}_q(u_n) \leq M$ and we prove the boundedness of $\|u_n\|_\infty$ thanks to (10) and the boundedness of $\|u_n'\|_2$ thanks to (11).

Step 2: $\exists\, n_k$ s.t. $u_{n_k} \to u$ in $W_n^{1,2}$.

Up to subsequence, by the previous point we get $u_n \rightharpoonup u$ for some $u \in W_n^{1,2}$. Since the embedding $W_n^{1,2} \hookrightarrow L^2$ is compact, we deduce, again up to a subsequence, that $u_n \to u$ in L^2. By (19), we have:

$$|\langle A(u_n), v\rangle - \langle f, v\rangle_{L^2}| \leq \varepsilon_n \quad \forall v \in W_n^{1,2}.$$

So, we can choose $v = u_n - u \in W_n^{1,2}$ and get

$$|\langle A(u_n), u - u_n\rangle| \leq \left|\langle \tilde{F}_q'(u_n), u - u_n\rangle\right| + |\langle f, u - u_n\rangle_{L^2}| \leq \varepsilon_n + \|f\|_2\|u_n - u\|_2.$$

Taking the limits on both sides of the previous inequality we get:

$$\lim_{n\to\infty} \langle A(u_n), u - u_n\rangle = 0.$$

Since the operator A is of type $(S)_+$, we conclude $u_n \to u$ in $W_n^{1,2}$.

■

We conclude the section by giving sufficient conditions in order to have at least three solutions for the elliptic PDE (2). We introduce

$$\bar{\mathcal{R}}\colon \mathbb{R} \to \mathbb{R}, \quad \bar{\mathcal{R}}(u) = \frac{1}{2}\int_{-1}^{1} \mathcal{R}(x,u)dx.$$

where we recall that $\mathcal{R}$ is such that:

$$\tilde{F}_q(u) = \frac{k}{2}\|u'\|_2^2 + \int_{-1}^{1} \mathcal{R}(x, u(x))\, dx.$$

The assumptions we need in order to get our results are basically three: (1) the space averaged EBM with potential $\bar{\mathcal{R}}$ has (at least) two stable steady-state solutions, (2) the viscosity $\kappa > 0$ of the 1D-EBM is sufficiently large, (3) the two wells in the potential functional $\bar{\mathcal{R}}$ corresponding to the two minimum points are sufficiently deep.

Theorem 18 *Assume $\bar{\mathcal{R}}$ has two minimum points $u_1 \neq u_2$, with $\tilde{F}_q(u_1) \geq \tilde{F}_q(u_2)$. There exist $\omega > 0$ and $f, g \in O(\varepsilon^{-1})$ as $\varepsilon \to 0^+$ s.t. if $\bar{\varepsilon} > 0$ satisfies:*

(i) $\bar{\mathcal{R}}''(u_i) > f(\bar{\varepsilon})$ *for* $i = 1, 2$,

(ii) $\kappa > g(\bar{\varepsilon})$,

(iii) $\bar{\varepsilon} \leq \omega$,

then $\tilde{F}_q$ has two local minimum points $\tilde{u}_1, \tilde{u}_2$ such that:

(a) $B_{H^1}(u_1, \bar{\varepsilon}) \cap B_{H^1}(u_2, \bar{\varepsilon}) = \emptyset$.

(b) $\tilde{u}_i \in B_{H^1}(u_i, \bar{\varepsilon})$, *for* $i = 1, 2$,

(c) If $\|u - u_1\|_{H^1} = \bar{\varepsilon}$, *then* $\tilde{F}_q(u) \geq \tilde{F}_q(u_1) + \delta$, *with* $\delta = \delta(\bar{\varepsilon}) > 0$.

Proof. The proof consists in repeating the direct method used to prove Proposition 7 and applying it to the set $H^1 \cap B_{H^1}(u_i, \bar{\varepsilon})$. Indeed, thanks to Lemma 19, we can find $\bar{\varepsilon} > 0$ s.t. $B_{H^1}(u_1, \bar{\varepsilon}) \cap B_{H^1}(u_2, \bar{\varepsilon}) = \emptyset$, $\tilde{u}_i \in B_{H^1}(u_i, \bar{\varepsilon})$ and

$$\|u - u_i\|_{H^1} = \bar{\varepsilon} \quad \Longrightarrow \quad \tilde{F}_q(u) - \tilde{F}_q(u_i) \geq \delta > 0, \quad \delta = \delta(\bar{\varepsilon}). \tag{20}$$

Now, we consider the set

$$\mathbb{X}_i = H^1 \cap \overline{B_{H^1}(u_i, \bar{\varepsilon})},$$

where we stress the fact that $\overline{B_{H^1}(u_i, \bar{\varepsilon})}$ denotes the closed ball in H^1. Considering a sequence $\{u_{n,i}\}_n$, we want to show that the sublevel sets of $\tilde{F}_q$ are compact in $\mathbb{X}_i$ under the following notion of convergence:

$$u_{n,i} \xrightarrow{\mathbb{X}_i} u_\infty \quad \text{if and only if } u_{n,i} \to u_\infty \text{ uniformly in } [-1,1] \text{ and } u'_{n,i} \rightharpoonup u'_\infty \text{ in } L^2.$$

Repeating the argument in the proof of Proposition 7, we get the existence of $u_{\infty,i} \in H^1$ s.t. $u_{n,i} \xrightarrow{\mathbb{X}_i} u_{\infty,i}$. Thanks to the uniform convergence, we have

$$\|u_{\infty,i} - u_i\|_{H^1}^2 = \|u_{\infty,i} - u_i\|_2^2 + \left\|u'_{\infty,i}\right\|_2^2 = \lim_n \|u_{n,i} - u_i\|_2^2 + \left\|u'_{\infty,i}\right\|_2^2.$$

Second, for each n it holds

$$\bar{\varepsilon}^2 \geq \|u_{n,i} - u_i\|_{H^1}^2 = \|u_{n,i} - u_i\|_2^2 + \left\|u'_{n,i}\right\|_2^2.$$

Then, using the inferior lower-semi-continuity of the norm, we get:

$$\begin{aligned}\bar{\varepsilon}^2 &\geq \liminf_n \left(\|u_{n,i}-u_i\|_2^2 + \|u'_{n,i}\|_2^2\right) = \|u_{\infty,i}-u_i\|_2^2 + \liminf_n \|u'_{n,i}\|_2^2 \\ &\geq \|u_{\infty,i}-u_i\|_2^2 \|u'_{\infty,i}\|_2^2 = \|u_{\infty,i}-u_i\|_{H^1}^2.\end{aligned}$$

Hence $u_{\infty,i} \in \overline{B_{H^1}(u_i,\bar{\varepsilon})}$. Since $\tilde{F}_q$ is lower semi-continuous, we get the existence of $\tilde{u_i}$ minimum point for $\tilde{F}_q$ in $\mathbb{X}_i$. But thanks to the property (20), we deduce $\|\tilde{u}_i - u_i\|_{H^1} < \bar{\varepsilon}$. Hence $\tilde{u}_i$ are local minimum points for $\tilde{F}_q$ in H^1.

■

Lemma 19 *Assume $\bar{\mathcal{R}}$ as a minimum point u_0. Then, there exists $\omega > 0$ and $f, g \in O(\epsilon^{-1})$ as $\epsilon \to 0^+$ s.t. if $\varepsilon > 0$ satisfies:*

(i) $\bar{\mathcal{R}}''(u_0) > f(\varepsilon)$,

(ii) $\kappa > g(\varepsilon)$,

(iii) $\varepsilon = \|u-u_0\|_{H^1} \leq \omega$,

then $\tilde{F}_q(u) \geq \tilde{F}_q(u_0) + \delta$, *with* $\delta = \delta(\varepsilon) > 0$.

Proof. Let $u \in H^1$ s.t. $\|u-u_0\|_{H^1} = \varepsilon$. Since $\varepsilon^2 = \|u-u_0\|_2^2 + \|u'\|_2^2$, we divide the proof in two cases according to the magnitude of $\|u-u_0\|_2^2$ and $\|u'\|_2^2$.

Case 1: $\|u'\|_2^2 \geq \varepsilon^2/2$ *and* $\|u-u_0\|_2^2 \leq \varepsilon^2/2$.

Since $\mathcal{R}(x,v)$ is locally Lipschitz in v uniformly in x, there exists $L = L(\mathcal{R},u_0) > 0$ and $\omega = \omega(\mathcal{R},u_0) > 0$ s.t.

$$|\mathcal{R}(x,v) - \mathcal{R}(x,u_0| \leq L|v-u_0|, \quad |u-u_0| \leq \omega,\ x \in [-1,1].$$

Thus, if $\|u-u_0\|_\infty \leq \|u-u_0\|_{H^1} = \varepsilon \leq \omega$, it holds

$$\int_{-1}^1 |\mathcal{R}(x,u(x) - \mathcal{R}(x,u_0)|dx \leq L\|u-u_0\|_1 \leq \sqrt{2}L\|u-u_0\|_2 \leq L\varepsilon$$

Using the previous inequality and the bound on $\|u'\|_2^2$, we have

$$\tilde{F}_q(u) - \tilde{F}_q(u_0) \geq -\left|\int_{-1}^1 \mathcal{R}(x,u(x)) - \mathcal{R}(x,u_0)\,dx\right| + \frac{k}{2}\|u'\|_2^2 \geq -L\varepsilon + \kappa\frac{\varepsilon^2}{4},$$

and thus $g(\varepsilon) := \frac{4L}{\varepsilon}$.

Case 2: $\|u'\|_2^2 \leq \varepsilon^2/2$ *and* $\|u-u_0\|_2^2 \geq \varepsilon^2/2$.

Let's consider $\bar{u} = \frac{1}{2}\int_{-1}^1 u(x)dx$. We start by pointing out two useful inequalities. First

$$|\bar{u}-u_0| = \left|\frac{1}{2}\int_{-1}^1 u(x)dx - u_0\right| = \left|\frac{1}{2}\int_{-1}^1 u(x) - u_0 dx\right| \leq \|u-u_0\|_\infty \tag{21}$$

Second, for each $x \in [-1,1]$, it holds:

$$|u(x) - \bar{u}| = \left| \frac{1}{2} \int_{-1}^{1} (u(x) - u(y))\, dx \right| \leq \frac{1}{2} \|u'\|_2 \cdot \int_{-1}^{1} |x-y|^{1/2} dy \leq \sqrt{2} \|u'\|_2, \tag{22}$$

where the first inequality follows from the Holder properties of the Sobolev space H^1. At this point, the estimate on the value of the functional at u can be done considering

$$\tilde{F}_q(u) - \tilde{F}_q(u_0) \geq \int_{-1}^{1} \mathcal{R}(x, u(x)) - \mathcal{R}(x, u_0) dx$$

and using a Taylor expansion for $\mathcal{R}$. Indeed,

$$\mathcal{R}(x, u(x)) = \mathcal{R}(x, u_0) + \mathcal{R}_u(x, u_0)(u(x) - u_0) + \mathcal{R}_{uu}(x, u_0) \frac{(u(x) - u_0)^2}{2} + O(\|u - u_0\|_\infty^3).$$

Performing the decomposition

$$u(x) - u_0 = (u(x) - \bar{u}) + (\bar{u} - u_0)\,,$$

we observe that:

$$\int_{-1}^{1} \mathcal{R}_u(x, u_0)(\bar{u} - u_0) dx = (\bar{u} - u_0) \int_{-1}^{1} \mathcal{R}_u(x, u_0) dx = (\bar{u} - u_0) 2 \bar{\mathcal{R}}'(u_0) = 0$$

and thus

$$\begin{aligned} \int_{-1}^{1} \mathcal{R}(x, u(x)) - \mathcal{R}(x, u_0) dx &= \int_{-1}^{1} \mathcal{R}_u(x, u_0)(u(x) - \bar{u}) dx \\ &+ \int_{-1}^{1} \mathcal{R}_{uu}(x, u_0) \frac{(u(x) - u_0)^2}{2} dx + O(\|u - u_0\|_\infty^3) \end{aligned} \tag{23}$$

The absolute value of the first term on the right-hand side can be bounded thanks to (22) and Holder's inequality. Indeed

$$\begin{aligned} \left| \int_{-1}^{1} \mathcal{R}_u(x, u_0)(u(x) - \bar{u}) dx \right| &\leq \|\mathcal{R}_u(\cdot, u_0)\|_\infty \|u - \bar{u}\|_1 \leq 2\sqrt{2} \|\mathcal{R}_u(\cdot, u_0)\|_\infty \|u'\|_2 \\ &\leq 2 \|\mathcal{R}_u(\cdot, u_0)\|_\infty \varepsilon. \end{aligned}$$

Now, need to estimate the second term on the RHS of (23). Adding and subtracting $\bar{\mathcal{R}}''(u_0)$, we have:

$$\begin{aligned} \int_{-1}^{1} \mathcal{R}_{uu}(x, u_0) \frac{(u(x) - u_0)^2}{2} dx &= \frac{1}{2} \int_{-1}^{1} \bar{\mathcal{R}}''(u_0)(u - u_0)^2 dx \\ &+ \frac{1}{2} \int_{-1}^{1} \left(\mathcal{R}_{uu}(x, u_0) - \bar{\mathcal{R}}''(u_0) \right) (u - u_0)^2 dx. \end{aligned} \tag{24}$$

The first term on the RHS is large thanks to the central assumption of Case 2. In fact

$$\frac{1}{2}\int_{-1}^{1}\bar{\mathcal{R}}''(u_0)(u-u_0)^2dx = \frac{\bar{\mathcal{R}}''(u_0)}{2}\|u-u_0\|_2^2 \geq \bar{\mathcal{R}}''(u_0)\frac{\varepsilon^2}{4}.$$

The absolute value of the second term on the RHS of (24) can be bounded using Holder's inequality as follows:

$$\left|\frac{1}{2}\int_{-1}^{1}\left(\mathcal{R}_{uu}(x,u_0)-\bar{\mathcal{R}}''(u_0)\right)(u-u_0)^2dx\right| \leq \frac{1}{2}\left\|\bar{\mathcal{R}}''(u_0)-\mathcal{R}_{uu}(\cdot,u_0)\right\|_\infty\|u-u_0\|_2^2$$
$$\leq \left\|\bar{\mathcal{R}}''(u_0)-\mathcal{R}_{uu}(\cdot,u_0)\right\|_\infty\frac{\varepsilon^2}{2}.$$

Summing up, we have shown

$$\tilde{F}_q(u)-\tilde{F}_q(u_0) \geq -2\varepsilon\|\mathcal{R}_u(\cdot,u_0)\|_\infty + \frac{\varepsilon^2}{4}\bar{\mathcal{R}}''(u_0) - \frac{\varepsilon^2}{2}\left\|\bar{\mathcal{R}}''(u_0)-\frac{\varepsilon^2}{2}\mathcal{R}_{uu}(\cdot,u_0)\right\|_\infty + O(\varepsilon^3),$$
$$= \frac{\varepsilon^2}{2}\bar{\mathcal{R}}''(u_0)+f_1(\varepsilon) =: \delta(\varepsilon)$$

with $f_1 \in O(\varepsilon)$, f_1 negative for small ε and

$$f(\varepsilon) := -\frac{2}{\varepsilon^2}f_1(\varepsilon) \in O(\varepsilon^{-1}).$$

■

In conclusion, applying the MPT we get the following result.

Corollary 20 *If the hypothesis of Theorem 18 are satisfied, then the elliptic problem* (2) *has at least three solutions.*

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