Referee report

Title: Variational Techniques for a One-Dimensional Energy Balance Model

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General comments

The paper studies an energy balance climate model which consists of a nonlinear parabolic equation in one space dimension (on the interval $[-1, 1]$) with Neumann boundary conditions. This is a well-known model that describes the evolution of Earth's temperature based on the planet's energy budget. The impact of carbon dioxide on the system is represented by an additive parameter $q \in \mathbb{R}$, which is treated as a bifurcation parameter. The authors adopt a variational approach to prove the existence of steady-state solutions and relate the structure oif the minimizing set to the differentiability of the associated value function.

The topic of the paper fits well to the schemes of the journal. The results are convincing and original: relating the uniqueness of the solution of the stationary problem with the smoothness of the value function is a very nice idea. Moreover, the paper is clearly structured and makes good usage of images to illustrate the described approach.

Specific comments

1. Modelling the effect of the CO_2 on the energy budget as the additive constant q is a choice that should be better motivated on physical grounds.

2. Removing the natural degeneracy of the diffusion coefficient at the boundary is more of a restriction than what the authors seem willing to admit on page 7. I understand that such a choice was made in order to reduce the complexity of the problem, but adapting the authors' approach to the real EBCM (degenerate parabolic equation) should be at least mentioned as an open problem.

3. In the context of optimal control, the value function is characterised as the solution of some nonlinear partial differential equation (the Hamilton-Jacobi equation). In this paper, it is shown that $-V'(q)$ equals the average of the minimizer of a certain functional. Could it be possible to characterize V as the solution of some equation?

Technical corrections

This paper is essentially typo-free. I only found one typo:

1. just above line 90: enable*s*