Reply on RC2

January 21, 2024

We would like to thank the reviewer for the careful reading and the suggestions to improve the manuscript. We have taken them into account, as indicated in the answer below.

- "Question 1. *Modelling the effect of the CO_2 on the energy budget as the additive constant q is a choice that should be better motivated on physical grounds.*"

  We agree with the reviewer about the necessity to better explain the choice to model the radiative forcing q of CO_2 as an additive term. We have motivated this choice by motivating it with a linearization argument involving the outgoing radiation R_e depending on temperature u and CO_2 concentration. The full argument can be found in Section 2.1 of the main manuscript.

- "Question 2. *Removing the natural degeneracy of the diffusion coefficient at the boundary is more of a restriction than what the authors seem willing to admit on page 7. I understand that such a choice was made in order to reduce the complexity of the problem, but adapting the authors' approach to the real EBCM (degenerate parabolic equation) should be at least mentioned as an open problem.*"

  The reviewer is right and we agree with him/her. The choice is made to be able to study with classical tools from calculus of variation the variational problem, from which all our results follow. We have clearly stated, in Section 2.1 (before the equation $\kappa(x) = D(1-x^2) + \delta$, $D, \delta > 0$) and in Section 4, that it remains an open problem how to extend our results when $\kappa = \kappa(x)$ is degenerate at the boundary of the spatial domain.

- "Question 3. *In the context of optimal control, the value function is characterised as the solution of some nonlinear partial differential equation (the Hamilton-Jacobi equation). In this paper, it is shown that $-V'(q)$ equals the average of the minimizer of a certain functional. Could it be possible to characterize V as the solution of some equation?*"

  We are not able to characterize the value function as the solution of an equation.

Indeed, continuing the parallel with optimal control, in that case the value function is, given an initial time and an initial state condition, the minimum value attained by the objective function. In our setting, the objective function corresponds to F_q. But F_q does not depend on initial time and initial state condition.

We have preferred not to add any digression about the Hamilton-Jacobi equation to the manuscript, to avoid inserting a topic not investigated in the work.