Supplementary Material.

Slow Manifold Reduction by Multiple-Scale Expansion

Begin with equations (2) and (3), written in the form

$$\frac{dx_i}{dt} = v_i \tag{13a}$$

$$\varepsilon \frac{dv}{dt} = \varepsilon \frac{3R}{2}\frac{Du}{Dt} + (v - u) + \varepsilon\left(1 - \frac{3R}{2}\right) g_r + 3\varepsilon R\Omega \times (u - v) + 2\varepsilon\left(\frac{3R}{2} - 1\right)\Omega \times v \tag{13b}$$

Let $\tilde{t} = (t - t_o)/\varepsilon$ represent a fast time variable, and consider the particle position and velocity to be functions of *t* and $\tilde{t}$, which are formally treated as separate variables. Thus d/dt is replaced by $\varepsilon^{-1}\partial/\partial\tilde{t} + \partial/\partial t$, where it is understood that both $\partial/\partial t$ and $\partial/\partial\tilde{t}$ are to be interpreted as particle-following derivatives (and not derivatives with position held constant). Note that the background flow does not depend on $\tau,$ and thus the substantial derivative $D\vec{u}/Dt$ is with respect to *t* alone. However, dependence on $\tilde{t}$ is introduced when the derivative is evaluated at the position $\vec{x}(\tilde{t}, t)$.

Expanding both variables in a power series in ε leads to

$$\vec{x} = \vec{x}^{(0)}(\tilde{t}, t) + \varepsilon\vec{x}^{(1)}(\tilde{t}, t) + \cdots$$

$$\vec{v} = \vec{v}^{(0)}(\tilde{t}, t) + \varepsilon\vec{v}^{(1)}(\tilde{t}, t) + \cdots$$

Substitution into (13) leads to

$$\left(\varepsilon^{-1}\frac{\partial\vec{x}^{(0)}}{\partial\tilde{t}} + \frac{\partial\vec{x}^{(0)}}{\partial t}\right) + \varepsilon\left(\varepsilon^{-1}\frac{\partial\vec{x}^{(1)}}{\partial\tilde{t}} + \frac{\partial\vec{x}^{(1)}}{\partial t}\right) + \cdots = \vec{v}^{(0)} + \varepsilon\vec{v}^{(1)} + \cdots, \tag{14a}$$

and

$$\varepsilon\left(\varepsilon^{-1}\frac{\partial\vec{v}^{(0)}}{\partial\tilde{t}}+\frac{\partial\vec{v}^{(0)}}{\partial t}\right)+\varepsilon^{2}\left(\varepsilon^{-1}\frac{\partial\vec{v}^{(1)}}{\partial\tilde{t}}+\frac{\partial\vec{v}^{(1)}}{\partial t}\right)+\cdots=\vec{u}\left(\vec{x}^{(0)}(\tilde{t},t),t\right)-\vec{v}^{(0)}+\varepsilon\frac{\partial\vec{u}}{\partial x_j}x_j^{(1)}$$

$$-\varepsilon v_i^{(1)}+\varepsilon[\frac{3R}{2}\frac{D\vec{u}}{Dt}+3R\vec{\Omega}\times\left(\vec{u}-\vec{v}^{(0)}\right)+2\left(\frac{3R}{2}-1\right)\vec{\Omega}\times\vec{v}^{(0)}+\left(1-\frac{3R}{2}\right)\vec{g}_r]+\cdots, \qquad (14b)$$

where again, the derivatives of $\vec{u}$ are evaluated at $\vec{x}^{(0)}$. To lowest order, we have

$$\frac{\partial\vec{x}^{(0)}}{\partial\tilde{t}}=0 \text{ and } \frac{\partial\vec{v}^{(0)}}{\partial\tilde{t}}=\vec{u}\left(\vec{x}^{(0)}(\tilde{t},t),t\right)-\vec{v}^{(0)}. \qquad (15a,b)$$

Thus $\vec{x}^{(0)}=\vec{x}^{(0)}(t)$, and since the right-hand side of (15b) is then independent of $\tilde{t}$, it follows that

$$\vec{v}^{(0)}=\vec{u}\left(\vec{x}^{(0)}(t),t\right)+\vec{c}^{(0)}(t)e^{-\tilde{t}}. \qquad (16)$$

If a particle is initiated with a velocity that is differs from the local fluid velocity by more than O(ε), then the drag on the particle brings it O(ε) close to the fluid velocity over a time scale of O(ε^{-1}). This behavior is consistent with the requirement in Fenichel theory that the background flow is a normally attracting manifold.

At the next order of approximation [O(*0*) in 14a], we have

$$\frac{\partial\vec{x}^{(1)}}{\partial\tilde{t}}=-\frac{\partial\vec{x}^{(0)}}{\partial t}+\vec{v}^{(0)}=-\frac{\partial\vec{x}^{(0)}}{\partial t}+\vec{u}\left(\vec{x}^{(0)}(t),t\right)+\vec{c}^{(0)}(t)e^{-\tilde{t}}$$

After decay of the final term, the remaining terms on the right-hand side depend only on t and therefore lead to secular growth in τ of $x_i^{(1)}$. To prevent this growth we must set these terms to zero:

$$\frac{\partial\vec{x}^{(0)}}{\partial t}=\vec{u}\left(\vec{x}^{(0)}(t),t\right) \qquad (17)$$

Which indicates simply that following the decay from the initial velocity, the particle follows the flow at leading order. Solving the remaining equation for $\vec{x}^{(1)}$ then gives

$$\vec{x}^{(1)} = \vec{x}_o^{(1)}(t) - \vec{c}^{(1)}(t)e^{-\tilde{t}} \tag{18}$$

Proceeding to O(ε) in (14b) then gives

$$\frac{\partial \vec{v}^{(1)}}{\partial \tilde{t}} + \vec{v}^{(1)} = -\frac{\partial \vec{v}^{(0)}}{\partial t} + \left(\frac{\partial \vec{u}}{\partial x_j}\right)_{x_i = x_i^{(0)}} x_j^{(1)} + \frac{3R}{2}\frac{D\vec{u}}{Dt} + 3R\vec{\Omega} \times (\vec{u} - \vec{v}^{(0)})$$

$$+2\left(\frac{3R}{2} - 1\right)\vec{\Omega} \times \vec{v}^{(0)} + \left(1 - \frac{3R}{2}\right)\vec{g}_r$$

Using (16) to substitute for $v_i^{(0)}$ on the right-hand side, and keeping in mind that $\frac{\partial}{\partial t}$ represents not a local time derivative but a time derivative with $\tilde{t}$ held constant, we have

$$\frac{\partial \vec{v}^{(0)}}{\partial t} = \frac{\partial}{\partial t}\vec{u}\big(\vec{x}^{(0)}(t), t\big) + \frac{\partial \vec{c}^{(0)}}{\partial t}e^{-\tilde{t}} = \frac{D\vec{u}}{Dt} + \frac{\partial \vec{c}^{(0)}}{\partial t}e^{-\tilde{t}}$$

Using this expression as well as (18) to substitute for and $x_j^{(1)}$ leads to, after some regrouping of terms, to

$$\frac{\partial \vec{v}^{(1)}}{\partial \tilde{t}} + \vec{v}^{(1)} = \vec{a}^{(1)}(t) - \vec{b}^{(1)}(t)e^{-\tilde{t}}, \tag{19}$$

where

$$\vec{a}^{(1)}(t) = \frac{\partial \vec{u}}{\partial x_j} x_{o,j}^{(1)}(t) + \left(\frac{3R}{2} - 1\right)\left[\frac{D\vec{u}}{Dt} - \vec{g}_r + 2\vec{\Omega} \times \vec{u}^{(0)}\right],$$

and

$$\vec{b}^{(1)}(t) = \frac{\partial c_i^{(0)}}{\partial t} + \frac{\partial \vec{u}}{\partial x_j} c_j^{(0)} + 3R\vec{\Omega} \times \vec{c}^{(0)}.$$

The solution to (19) is

$$\vec{v}^{(1)} = \vec{a}^{(1)}(t) - \vec{b}^{(1)}(t)\tilde{t}e^{-\tilde{t}} \quad (20)$$

We can now write down an expression for the particle velocity on the slow manifold, obtained by taking the limit $\tilde{t} \to \infty$ in (16) and (20):

$$\vec{v}^{(0)} + \varepsilon\vec{v}^{(1)} = \vec{u}\left(\vec{x}^{(0)}(t), t\right) + \varepsilon \frac{\partial \vec{u}}{\partial x_j} x_{o,j}^{(1)}(t) + \varepsilon\left(\frac{3R}{2} - 1\right)\left[\frac{D\vec{u}}{Dt} - \vec{g}_r + 2\vec{\Omega} \times \vec{u}^{(0)}\right]$$

or, noting that $\vec{x} = \vec{x}^{(0)} + \varepsilon\vec{x}_o^{(1)}(t) + O(\varepsilon^2)$ on the slow manifold:

$$\frac{d\vec{x}}{dt} = \vec{u}(\vec{x}(t), t) + \varepsilon\left(\frac{3R}{2} - 1\right)\left[\frac{D\vec{u}}{Dt} - \vec{g}_r + 2\vec{\Omega} \times \vec{u}^{(0)}\right] + O(\varepsilon^2). \quad (2$$