Referee 3 (Report 1 15 May 2024)

We are grateful to the referee for devoting time to our manuscript.
We will here respond to comments made:

(Major Comment A) Concerning the physical relevance of the model, the qualitative features such as patterns, phase, and group velocities within the system fail to convince, primarily because the equation omits mechanisms like potential vorticity (PV) generation necessary for planetary waves (Rossby waves). To address this issue, it is advisable to include a disclaimer stating: "Although the equation lacks the forcing terms essential for generating planetary waves, it qualitatively mimics features similar to those of the Rossby wave."

Response: We have added the information that planetary waves are not generated by potential vorticity. (Lines 717-721 in revised manuscript):

To a certain extent, the model quantitatively describes weather systems, but unlike the well-known Lorenz model of atmospheric convection (Lorenz, 1963), it cannot be derived from any atmospheric dynamic equations. The motivation was to formulate the simplest possible set of dissipative chaotically behaving differential equations that share some properties with the "real" atmosphere. Although mechanisms such as potential vorticity generation are lacking in the equations, the model generates 5 to 7 main highs and lows corresponding to planetary waves (Rossby waves). To keep 5 to 7 main highs and lows ~~that correspond to planetary waves (Rossby waves)~~, Lorenz (2005) suggested a ratio $N/L = 30$ and $F = 15$.

(Major Comment B) Regarding the impact of small-scale processes, this study and the existing literature identify two types of such processes.

Type (1) involves small-scale processes introduced through an increased number of grid points or spectral modes. In this study, models with a larger number N incorporate smaller scale processes. However, as shown in Table RR1, increasing N reduces the magnitude of the Lyapunov exponent for smaller values of N. For larger N values (N=150, 360, 960), the Lyapunov exponents remain the same (but why?), indicating that the inclusion of smaller scale processes does not enhance instability.

In contrast, type (2) involves small scale processes through model coupling. The reviewer notes that not only do the newly introduced small scale processes but also the coupling methodology potentially impact the system's stability. [It should be noted that in real-world models, smaller spatial and temporal scale processes associated with parameterizations typically fall into this category.]

Indeed, the reviewer has previously formulated a set of generalized Lorenz models demonstrating several key insights: (1) incorporating spectral modes and additional dissipative terms at higher wavenumbers can lead to systems that only exhibit chaotic behaviors at higher critical values of the Rayleigh parameter (e.g., Shen 2019); (2) integrating smaller-scale heating processes could lead to system destabilization (e.g., Shen 2015); (3) selecting specific spectral models, akin to model coupling, can significantly influence the stability of the system.

This is why the reviewer recommended investigating (i) the impact of the coupling coefficients (ii) the potential role of nonlinear terms in generating additional critical points. Regrettably, these comment have not been addressed adequately. To remedy this, it is advised to acknowledge the following: (a) the two types of small scale processes that may generate different feedback effects, and (b) the role of coupling as another influential factor in determining the stability of the coupled system and the true impact of small scale processes on stability.

(As a result, the following statement in Abstract is not accurate: When studying the initial error growth, it turns out that small scale phenomena, which contribute little to the forecast product, significantly affect the ability to predict this product.)

Response: For the L05-1 system, it is not valid that increasing N allows the involvement of smaller scale processes. The reason is that for the L05-1 system, there is an attempt to keep 6-7 main waves and several smaller waves through the linking of X_n variables. For a smaller number of variables N, the smaller waves are more pronounced and therefore the value of the Lyapunov exponent is larger for smaller N. For larger N, the ratio of major and minor waves is similar and therefore the value of the Lyapunov exponent remains the same for higher N.

Smaller scales are added to L05 systems using the procedure described on lines 731-764 (in revised manuscript):

Lorenz (2005) wanted to keep the system as simple as possible, so instead of, for example, Fourier analysis, a procedure for expressing variables $X_{tot,n}$ as sums of $X_{1,n}$ and $X_{2,n}$ was introduced:

$$X_{1,n} = \sum_{i=-I}^{I}{}' \left(\alpha - \omega|i|\right) X_{tot,n+i}, \tag{A1}$$

$$X_{2,n} = X_{tot,n} - X_{1,n}. \tag{A2}$$

Parameters α, ω, and I are chosen so that X_1 is a low-pass filtered version of X_{tot}, and X_2 represents the difference between the full signal X_{tot} and the filtered signal. By this procedure, X_2 has a much smaller amplitude than X_1, and also its time evolution should be faster since the temporal derivative is related to the spatial derivative via the difference $(X_{1,n+1} - X_{1,n-2})$, which for the low pass filtered signal X_1 typically is smaller than for the signal X_2.

More precisely, Lorenz's (2005) idea is that the parameters α, ω are chosen so that X_1 equals X_{tot} whenever X_{tot} changes quadratically over the longitudes (variables) $n - I$ through $n + I$. It is when $\sum_{i=-I}^{I}{}'\left(\alpha - \omega|i|\right) = 1$ and $\sum_{i=-I}^{I}{}' i^2 \left(\alpha - \beta|i|\right) = 0$. By solving these equations, we get:

$$\alpha = \left(3I^2 + 3\right) / \left(2I^3 + 4I\right), \tag{A3}$$

$$\omega = \left(2I^2 + 1\right) / \left(I^4 + 2I^2\right). \tag{A4}$$

The procedures (Eqs. (A4) and (A5)) are functions of the interval length $[-I, I]$.

When creating a system dX_{tot}/dt as the sum of dX_1/dt and dX_2/dt (sum of Eqs. (A2) and (A3)), the coupling term $cX_{1,n}$ in Eq. (A3), which enables short waves to develop, is combined with the dissipation term $-X_{1,n}$ in Eq. (A2). Therefore, the coupling term can be canceled entirely, or it can appear in X_1 rather than X_2 when X_{tot} is analyzed, and there might be nothing to enable the short waves in X_2 to grow. Lorenz (2005) reformulated the coupling process by adding a small fraction of X_1 to X_2 so small waves in X_2 can amplify. It is done by replacing $b^2[X_2, X_2]_{1,n} + cX_{1,n}$ by $[X_2, X_2 + c'X_1]_{1,n}$ in Eq. (A3) , and L05-2 system would be:

$$dX_{tot,n}/dt = [X_1, X_1]_{L,n} + b^2[X_2, X_2]_{1,n} + c[X_2, X_1]_{1,n} - X_{1,n} - bX_{2,n} + F, \tag{A5}$$

where $c = c' \cdot b^2$.

Based on the L05-2 system (Eqs. (A4) - (A8)), Bednar and Kantz (2022) designed a three levels (scales) system (L05-3):

$$dX_{tot,n}/dt = [X_1, X_1]_{L,n} + b_1^2[X_2, X_2]_{1,n} + b_2^2[X_3, X_3]_{1,n} + c_1[X_2, X_1]_{1,n} + c_2[X_3, X_2]_{1,n} - X_{1,n} - b_1X_{2,n} - b_2X_{3,n} + F, \tag{A6}$$

where c_1, c_2, b_1, b_2 are parameters, and the procedure for expressing the variables are:

$$X_{1,n} = \sum_{i=-I_1}^{I_1}{}' \left(\left((3I_1^2 + 3)/(2I_1^3 + 4I_1) \right) - \left((2I_1^2 + 1)/(I_1^4 + 2I_1^2) \right) |i| \right) X_{tot,n+i}, \tag{A7}$$

$$X_{2,n} = \sum_{j=-I_2}^{I_2}{}' \left(\left((3I_2^2 + 3)/(2I_2^3 + 4I_2) \right) - \left((2I_2^2 + 1)/(I_2^4 + 2I_2^2) \right) |j| \right) \left(X_{tot,n+j} - X_{1,n+j} \right), \tag{A8}$$

$$X_{3,n} = X_{tot,n} - X_{2,n} - X_{1,n}, \tag{A9}$$

where I_1 and I_2 set the length of the intervals $[-I, I]$.

In our case, coupling refers to the linking of different scales that allows the formation of smaller waves. It should be noted that the smaller scales are filtered out of the overall X_{tot} variable by the method described by equations A4-A5 for the L05-2 system and A10-A12 for the L05-3 system, and thus this is more of a reviewer's type of process (1). The processes described in Shen (2019) and Shen (2015) would then be comparable to adding another variable that affects the variable X_{tot}, but which is not filtered out of X_{tot}. For example, if we define the constant F in L05 systems as a variable and describe it by its own ordinary differential equation dF/dt. Depending on the definition of dF/dt, we could then obtain phenomena similar to those described in Shen (2019) and Shen (2015).

It is shown in Bednar and Kantz (2022) that in the power law $dEp/dt = aE^{(1-b)}$, the coupling rate (as defined in L05 systems) is described by the value of the parameter a and does not affect the value of b. From this we conclude the general validity of the published results for different coupling rates.

We agree that the statement in the abstract may not always be valid in general and have therefore modified it. (Lines 8-9 in revised manuscript):

When studying the initial error growth, it may turn~~s~~ out that small scale phenomena, which contribute little to the forecast product, significantly affect the ability to predict this product.

(Major Comment C) This research commenced by presuming the existence of a Lyapunov exponent (LE) and proceeded to formulate various ODEs aimed at determining the predictability horizons, emphasizing the impact of (reducing) initial conditions. However, the reviewer wishes to highlight several crucial points: (1) the LE signifies a time-averaged measure; (2) all the ODEs (with b > 0) examined in this study maintain continuous dependence on initial conditions (CDIC) across infinite time intervals, which precludes them from disclosing finite predictability for chaotic solutions. Additional insights and related discussions, which are available in Shen (2024), are provided below:

(1) For a linear ODE given by E' = sigma E, the solution E grows unboundedly. Predictability horizons can be extended by reducing initial errors or elevating thresholds.

(2) In the Logistic ODE, the solution E is bounded. However, the zero state (backward in time) and saturation value (forward in time) are only asymptotically reachable, implying that predictability horizons may also be extended by reducing initial errors or raising thresholds.

(3) For the major ODE in this study, E' = a E^{1-b} with a > 0 and Eo > 0, its solution is written as follows:

E = (Eo^b + abt)^{1/b}

Please note that the above solution is (1) unbounded for b > 0 and (2) unbounded within a finite time interval (i.e., Tmax = - Eo^b / (ab)) for b < 0 and finite Eo. It appears t have been considered in this study. For instance, Equation (9) becomes invalid for b < 0.

The author appreciates the discussion using the above solution that initial growth rates might not always be exponential. However, such formulations do not yield reliable pred horizons over longer periods. More importantly, what is the relationship between the parameter "a" and the Lyapunov exponent? The same inquiry applies to the Logistic ODE: between the parameter "sigma" and the Lyapunov exponent?

Response: The authors and most certainly also the referee understand that the mathematical concept of the Lyapunov exponent is of limited value for the growth of forecast errors starting from finite (i.e., non-infinitesimal) perturbations and having a finite extension of the attractor. Therefore, empirical/numerical error growth is always limited to growing not beyond a finite value, and it might show behaviors different from an exponential growth also in the initial growth phase, in which case the relation to the value of the mathematically defined Lyapunov exponent of the system cannot be exact.

Our definition of the predictability horizon is the time when the mean error growth reaches 95% of a saturation value E_{lim}. The intrinsic predictability is then this time when the average error of initial conditions (ie(0)) goes limitingly to zero. The intrinsic predictability horizon for scale-dependent error growth was determined using an **extended** power law (Eq. (4) in the revised manuscript), where the values of the parameters a,b, E_{lim} were determined from an approximation of the data and at $E_{ie}(0)$ -> 0.

The "sigma" parameter in the logistic ODE approximates the largest Lyapunov exponent for scale-independent error growth. For the power law (Eq. (3) in revised manuscript) and the extended power law (Eq. (4) in revised manuscript), when describing scale-dependent error growth, the parameter *b* is related to the largest Lyapunov exponents of each scale. The parameter a describes the degree of coupling of each scale. A detailed discussion and meaning of the parameters a, b can be found in Bednar and Kantz (2022) in Section 3.3.1

(Major Comment D) While Figure RR4 suggests the smallest growth rate for the L05-1 system and the largest for the L05-3 system, this appears inconsistent with the data from Figures 5-7, show 0.46 for the L05-1, L05-2, and L05-3 systems, respectively. Could you provide any explanations for this discrepancy?

Response: For finite perturbations, where the initial error growth is not perfectly exponential and later some saturation occurs, there is no uniquely defined error growth rate, but only fitted values of the parameters of the different error growth laws.

Based on the reviewer's reported value of 0.46, we assume that the reviewer is considering the value of the error growth rate for exponential growth (lambda$_{ex}$) reported for the L05-3 system. Fig. 7 shows the value lambda$_{ex}$ = 0.46 1/day. (Lines 843-846 in revised manuscript)

the early part of the growth by integration of dE_{ex} (E_{ex} , green, dashed) with $\lambda_{ex} = 0.46$ 1/day, integration of dE_r (E_r , blue, dashed) with $\lambda_r = 0.35$ 1/day and $\beta_r = 0.07$ unit/day, integrations of dE_p (E_p , red, dashed) with $a = 0.37$ unit$^{0.63}$/day and $b = 0.63$ and approximation of the full curve by integration of dE_{qv} (E_{qv} , green) with $\lambda_{qv} = 0.2$ 1/day and $E_{\lim} = 6.9$ unit, integration of dE_q (E_q , blue) with $\lambda_q = 0.14$ 1/day, $\beta_q = 0.17$ unit/day and $E_{\lim} = 6.9$ unit

Fig. 5 shows the value lambda$_{ex}$ = 0.33 1/day. (Lines 818-821 in revised manuscript)

the early part of the growth by integration of dE_{ex} (E_{ex} , green, dashed) with $\lambda_{ex} = 0.33$ 1/day, integration of dE_r (E_r , blue, dashed) with $\lambda_r = 0.32$ 1/day and $\beta_r = 0.00006$ unit/day, integrations of dE_p (E_p , red, dashed) with $a = 0.34$ unit$^{0.02}$/day and $b = 0.02$ and approximation of the full curve by integration of dE_{qv} (E_{qv} , green) with $\lambda_{qv} = 0.32$ 1/day and $E_{\lim} = 8.1$ unit, integration of dE_q (E_q , blue) with $\lambda_q = 0.32$ 1/day, $\beta_q = 0.003$ unit/day and $E_{\lim} = 8.1$ unit

Fig. 6 shows the value lambda$_{ex}$ = 0.29 1/day. (Lines 829-833 in revised manuscript)

the early part of the growth by integration of dE_{ex} (E_{ex} , green, dashed) with $\lambda_{ex} = 0.29$ 1/day, integration of dE_r (E_r , blue, dashed) with $\lambda_r = 0.26$ 1/day and $\beta_r = 0.02$ unit/day, integrations of dE_p (E_p , red, dashed) with $a = 0.25$ unit$^{0.32}$/day and $b = 0.32$ and approximation of the full curve by integration of dE_{qv} (E_{qv} , green) with $\lambda_{qv} = 0.2$ 1/day and $E_{\lim} = 6.8$ unit, integration of dE_q (E_q , blue) with $\lambda_q = 0.18$ 1/day, $\beta_q = 0.05$ unit/day and $E_{\lim} = 6.8$ unit

It can be seen that the lambda$_{ex}$ value is not the same for the L05-1, L05-2 and L05-3 systems. It is true that the lambda$_{ex}$ value is lowest for the L05-2 system and not for the L05-1 system, but this is because lambda$_{ex}$ is not a suitable indicator for scale-dependent error growth.

References:

Bednář, H., and Kantz, H.: Prediction error growth in a more realistic atmospheric toy model with three spatiotemporal scales, Geosci. Model Dev., 15, 4147–4161, https://doi.org/10.5194/gmd-15-4147-2022, 2022.