Response to Reviewer #1

We would like to thank reviewer #1 for their thoughtful comments on the new revision of the manuscript. We begin by responding to the major comments given:

1. The first major comment discusses whether the outcome of the RTS smoother estimate truly restores conservation rules that were violated by a Kalman filter estimate. The reference to Bryson and Ho (1975),and in particular Problem 3 on page 395, was helpful. We outline below how this problem proves that the RTS estimate restores violations in conservation laws introduced by the KF.

   Bryson & Ho (1975; in the following BH75) lay out the solution to the KS problem in three steps (sections 13.1–13.4), showing that updates to the initial conditions and external forcings, obtained recursively, are then used as inputs to the free-running model:
   (1) Considering a single-state transition, BH75 show how a correction to the state at time $t = 0$ (initial condition) and external forcing are obtained using observations at time $t = 1$.
   (ii) Extending this to a multi-stage process, the initial conditions subject to future observations and forcing updates at times $t = i$ are obtained recursively (eqns. 13.2.1–13.2.4), but requiring non-trivial computations in practice (rendering an exact implementation of the KS difficult for complex applications).
   (iii) The general smoothing process of a nonlinear dynamical system is sketched in BH75, section 13.4. Again, upon computing smoother updates to the forcing and initial condition, the state evolution is obtained via the direct integration of the forward model, thus fulfilling all conservation or invariance principles imposed by the underlying governing equations.

   The issue is briefly addressed in the manuscript in lines 302–307, and a reference to Bryson and Ho has been added.

2. The second major comment was in regards to labelling and outlining the different experiments described in the text. As the reviewer notes, we made an effort towards this with Table 1, and do feel that this does a good job of describing the parameters chosen for each experiment conducted with the mass-spring oscillator toy problem. Labelling the experiments with designated names is useful is certain circumstances, but here the experiment discussions are largely self contained, so references to them outside of their respective sections can be best done via a section reference itself.

   Regarding adding a table for the Rossby wave experiments, this does not seem necessary as only one experiment setup is actually considered. We attempted to clearly outline the setup for the truth and prediction in Section 4.1.1., before beginning discussions on energy estimates and WBC estimates. Once the prediction setup is known, alongside the noise that was added to the data points and the location of data, the KF and RTS assumptions follow. We run the KF to find the prediction $\tilde{\mathbf{x}}(t)$, and subsequently the RTS smoother $\tilde{\mathbf{x}}(t, +)$, both only once. From these modal estimates we compute we find the energy and WBC estimates.

3. Major comment 3 was with regards to our choice of $L = a$ where a is the radius of the Earth. We chose $L = a$ because the width of the Pacific is the same order of magnitude as the radius of the Earth. It is coincidental but in the spirit of a toy model, it does eliminate one more non-essential parameter from the system. (It does not change the physical character of the solution.)

4. Major comment 4 was in regards to Eq. (29), the approximate solution to the Stommel equation. Namely, it was suggested to divide all terms on the RHS by β'. This solution is taken from Pedlosky (1965) (Eq. 5.11), and we think it is correct as is.

5. The final major comment was in regards to the length of the discussion section, and that the manuscript lacks a conclusion or summary section. We have relabelled the Discussion section to be Discussion and Conclusions, as we intended for this section to conclude the manuscript. We have also expanded the end of the Conclusions section slightly in response to comments by Reviewer #2.

With regards to the rigorous, "specific points" about typos and suggested rewrites, all have been reviewed and applied where necessary in our effort improve the manuscript. These were very helpful and greatly appreciated. Below we respond to several of them directly. We have reproduced the reviewer's comments in *italic*:

L. 205: Please elaborate.

We have added the following sentence:
"The uncertainty in the predicted displacement is small once new data are inserted via the analysis increment, but it quickly grows as the model is integrated beyond the time of analysis."

Fig. 13, caption, "... slow increase towards a higher value when no data are available.": This is not apparent in figure 13b.

To clarify our statement, we have added the following sentence in the caption:
"(we note here that $\mathbf{Q}(t,+)$ is computed from $t = T_f$ backwards to $t = 0$)."

L. 482, "... observations are not independent of the state vector...": Please drop or rephrase.

We have rephrased as follows:
"... observations are derived quantities of the state vector..."

L. 501-502: I do not see the rationale for introducing the two distribution matrices, Γ *and* $\mathbf{B}$. *It seems to me that* Γ *and* $\mathbf{B}$ *are always the same matrix (e.g., Bryson and Ho 1975).*

In general, $\mathbf{B}$ distributes the known part of forcing, whereas Γ distributes the unknown part of the forcing or controls, as in the Rossby wave example with $\mathbf{u}(t)$. In the mass-spring oscillator example the KF estimates were only forced by $0.5 \cdot \mathbf{q}(t)$, and there was no knowledge of the white noise component $\varepsilon(t)$. This is an example where there exists an unknown forcing that would be distributed by Γ. The reviewer is correct that here $\mathbf{B} \equiv \Gamma$, but we could have designed the experiment where only mass three (for one example) was forced by a white noise component, which forces Γ to be a different operator than $\mathbf{B}$.

L. 523, "innovation equation": Please identify or report this equation in the manuscript.

We identify eqn. (4) as the innovation equation.

L. 551-552, "... depend upon ... and with many observations including those ...": Please rephrase.

We have changed the sentence as follows:
"The various time-scales embedded in $\mathbf{L}$ depend upon those in $\mathbf{A}, \mathbf{P}(t,-), \mathbf{P}(t)$, and upon the observations entering the KF, including the observational sampling intervals and the structure in the observational noise."

References

Bryson, A. E. and Ho, Y.-C.: Applied optimal control, revised printing, Hemisphere, New York, 1975.
Pedlosky, J.: A study of the time dependent ocean circulation, Journal of Atmospheric Sciences, 22, 267–272, 1965.