Response to RC1: 'Comment on egusphere-2023-1252', Anonymous Referee #1, 08 Aug 2023'

January 30, 2024

Reviewer Comment: Determining the fate of surface and basal cracks on ice shelves is an important yet unsolved problem in the cryosphere. In this paper, following a prior work focused on the development of the boundary element model and LEFM by the same authors, a novel dynamical system approach is employed to generalize the predicted outcomes of surface and basal cracks for a range of relevant parameters. The final result is subject to a few assumptions, such as the periodic boundary condition, and the neglect of the buoyancy restoring force related to the ice-ocean boundary condition. However, the dynamical system analysis and interpretation of the phase space are generally applicable pending improvements to the LEFM model itself. My comments primarily pertain to the clarity of the model explanations. With these minor revisions, I recommend this paper for publication.

First, as this is the first time a phase plane approach for crack propagation is presented, the authors can include some examples where clear theoretical expectations exist to facilitate the explanation of the phase plane. For example, the authors mention that their previous analytical result that $\tau = 0.039$ would cause calving by basal crevasse. Theoretically without surface water we would simply see zero surface crack growth and basal crack growth. I think Fig 3 a4 attempted to illustrate this, but I'd think for $\eta = 0.1$, any surface crack depth < 0.1 corresponds to dry surface crack. Thus $d_t < 0.1$ would all correspond to arrows pointing towards the right. Could the authors explain the curved trajectories near $[dt, db] = [0, 0]$?

Response: If we understand the comment correctly, then it appears to say is that a dry surface crack should not be able to propagate. That would, however be incorrect: consulting figure 6a of Zarrinderakht et al (2022; cited in the present manuscript), we can see that at $\tau = 0.04$, there is a finite region of non-steady surface cracks lengths near $d_t = 0$ even in the absence of basal cracks, between the dashed and solid maroon lines in that figure. Physically, these cracks grow because for relatively short cracks the extensional stress τ acting on the crack faces is able to overcome the effect of cryostatic pressure and generate a positive stress intensity factor even when the crack is dry.[1]

[1] A minimal crack length is however required to elevate that intensity factor to or above the fracture toughness; that short length is too small to be rendered in the phase diagrams of the present paper, but corresponds to the region between the horizontal axis and the dashed maroon line of figure 6a of Zarrinderakht et al 2022. We refer to this small region in the present paper when we say in section 3.1,

The last caveat arises because, for the small values of κ relevant to typical ice shelves, there is a small region around the origin in the phase plane (not visible in Figure 2 due to its small size) in which neither crack will grow. When we state that orbits started near the origin will evolve towards the node, we have to add that they need to start outside that small region. The existence of short steady-state crack lengths has been discussed previously (van der veen 1998a, Lai et al 2020), and is associated with low-stress intensity factors, scaling as $d^{1/2}\tau$ for the surface crack, and $d^{1/2}(\tau - 1 + r^{-1})$ for the basal crack (see appendix C1 of Zarrinderakht et al, 2022): for small enough d, these

Of course, make the dry surface crack too long, however, and cryostatic pressure will cause the stress intensity factor to go to zero. These observations would account for why orbits along the d_t-axis point up for fairly small $d_t \lesssim 0.1$ in figure 3a4 of the present submission, in line with figures 2a and 6a of Zarrinderakht et al (2022).

We have amended the text in two places to tie these concepts together. First, at the end of the revised section 2, we expand on phase place plotting by linking the information contained in phase plane plots with the simpler ways of graphically displaying how cracks grow in previous papers on single cracks:

Plotting orbits on a phase plane provides a simple graphical way of identifying the behavior of the system for a given set of parameters, for all possible initial conditions and a given set of parameter values. In that way, a phase plane is analogous to for instance to Figure 10 of van der Veen(1998a), or Figures 4 and 7 in Zarrinderakht et al (2022) for the single-crack systems considered in these papers. There, the evolution of the single crack length variable d is determined graphically by plotting $K_{\mathrm{I}}(d)$*: from this, one can read off whether a crack lengthens or not depending on whether or not stress intensity factor exceeds fracture toughness* ($K_{\mathrm{I}} > \kappa$) *or not. A perhaps even more direct analogue to a phase plane is shown in Figures 6 and 8b of Zarrinderakht et al (2022), where (for a given set of parameter values), the range* $0 \leq d \leq 1$ *is divided into intervals for which* $\dot{d} = 0$ *and* $\dot{d} > 0$*, and hence indicates what state d evolves. A phase plane generalizes this by not only indicating where* $\dot{d}_{\mathrm{b}}$ *and* $\dot{d}_{\mathrm{t}}$ *are positive and zero, respectively, but by showing the relative size of the rates of change, which determines the angle of the orbit and ultimately the state that the cracks evolve towards.*

To address the "clear theoretical expectations" part of the point raised by the reviewer (modulo the explanation of why the clear theoretical expectations may differ from those the reviewer has in mind), we have amended the end of section 3.1 to address specifically the behaviour on (or very near) the coordinate axes of the phase plane, pointing out that the dynamics of the system along the coordinate axes is exactly that predicted by previous studies of single cracks, but also making clear that predictions of stability and instability for those single cracks may fail when coupled with another crack:

The third marginal fixed point is marked with a green dot in Figure 2 and again plays the role of a saddle in a standard phase plane. Here, one orbit (marked in green) emerges from the saddle point towards the calving boundary while a second orbit (also marked in green) connects a fourth (cyan) marginal fixed point that is almost on the d_{t}*-axis to the green saddle point; this orbit is analogous to a separatrix in a standard phase plane, and divides initial conditions that lead to immediate calving from initial conditions that lead to stable, steady cracks of finite length that leave the ice slab intact. The cyan-coloured unstable node point is paired with a third saddle point marked in magenta that is also located almost on the* d_{t} *axis ("almost" because there is in fact an imperceptible region near the* d_{t}*-axis in which* d_{b} *does not grow, as discussed above). For the dynamics of a single top crack, the magenta and cyan points were previously identified elsewhere (see e.g., figures 2a and 6a,b of Zarrinderakht et al 2022) as stable and unstable equilibria, evolution of short cracks towards the smaller (magenta) of the two being physically explained by the effect of the imposed tensile stress* τ *in opening the short crack eventually being overcome by cryostatic pressure as the crack lengthens. In fact, in all of the phase planes shown in the paper, the dynamics along each of the coordinate axes reduces to the dynamics of a single as previously discussed in Lai et al (2020) and Zarrinderakht et al (2022). As figure 2 shows, the simple rationale regarding the stability of single cracks developed previously in these papers however falters when coupling the surface crack with a basal crack: for instance, both the magenta and cyan equilibria are in fact unstable to the growth of a basal crack.*

are guaranteed to be less than fracture toughness κ*.*

Reiewer Comment: Would you have a simpler phase plane if water height is rescaled by the crack depth rather than ice thickness?

Response: Assuming that scaled water height is to be treated as a constant, then scaling the physical water height by crack depth corresponds to different physics (a water table whose location changes as the crack evolves, in such a way that the water table is always at a certain fraction of the crack length. It's possible this would elad to a simpler phase plane, depending on the definition of "simpler". However, we struggle to envisage a physical situation in which the water table would evolve in this way. A perhaps easier to justify alternative to our fixed water table elevation would be a fixed water volume in the crack. As discussed in Zarrinderakht et al (2022), that situation however comes with a number of awkward caveats, and as a result, we have not considered it here.

Reviwer Comment: On the other hand, when the surface crack is fully filled with water, and no resistive stress $\tau = 0$, it makes sense that surface crack always leads to full-depth penetration (figure 3e1). However, why by increasing the resistive stress to $\tau = 0.04$ (figure 3a1) the surface crack $d_t < 0.15$ would not propagate? This seems to have an important implication, that when the ice surface is fully filled with meltwater, as long as the resistive stress is large enough and the surface crack under certain depth, calving could be induced by basal crack rather than surface crack.

Response: We believe that figure 3a1 does show the surface crack d_t propagating for all values d_t above an extremely narrow and visually unresolved strip along the d_b axis[2] The only difference from figure 3e1 is that the basal crack *also* propagates[3] As a result, the orbits shown are no longer vertically but diagonally upwards. No matter where we look in figure 3a1, the orbit are at least a finite angle with the horizontal (if note vertical), indicating that the top crack d_t (plotted along the vertical axis) is growing. It is possible that we have misunderstood the comment, however.

Reviewer comment: Second, in the propagation rate model described in equation 4, the crack reaches steady state when the stress intensity factor reaches the fracture toughness. As demonstrated in figure 6, generally when the stress intensity factor curve bends back down at larger crack depth, there exists two steady state crack depths. Are the $K = K_c$ steady states at the smaller d_t, d_b simply treated as unstable steady states in the dynamical system approach? Can the author specify this?

Response: it's probably not so much that we treat them as unstable steady states as that they are unstable by the usual definition of stability (in the sense that they are *not* Lyapunov stable. We have expanded on this by extending (and splitting) the sixth paragraph of section 3.1 of the original submission, saying

The usual notions of phase plane analysis, like identifying isolated fixed points and their stability, do not apply without modification due to the non-differentiability of the dynamical system, and due to the fact that equilibria occupy extended regions of the phase plane. Equilibria inside these extended regions are stable in the sense of Lyapunov but not asymptotically stable (Strogatz, 1994): if perturbed, the state variable $(d_\mathrm{b}, d_\mathrm{t})$ *stays nearby because it does not evolve. For equilibria on the boundary of a region of steady states (that is, equilibria on one of the marginal nullclines), we can distinguish between unstable and stable. The boundary is unstable there are orbits that point away from it, which is the case for boundaries at the top or to the right of a region of steady states, and stable (again in the sense of Lyapunov) otherwise.*

There are equilibria that occupy a special role, namely those where two marginal nullclines intersect. We will refer to these equilibria as marginal fixed points below. [...]

[2] This is again the strip in which the surface crack d_t is too short for the stress intensity factor to rise above K_{Ic}, even with the water table at the surface.

[3] This is true at least until d_t reaches some appreciable size, at which point presumably the torques generated by the top crack prevent the bottom crack from growing further.

Reviewer comment: Finally, please note that the left hand side of equation 4 is still dimensional; the units on the left-hand side don't seem to align with the unit on the right-hand side. What other system parameters are missing to ensure a matching unit and determine the time scale of crack propagation?

Response: The units of the derivative of the "universal function K" in the denominator fix thus problem (see Zarrinderakht et al 2022, eqs (18)–(19), or better still, the original derivation in Freund's (1990, chapoter 6) book. The universal function K itself has no units (see equation (6.4.26) and the discussion in the subsequent paragraph of Frenud (1990) as well as equation 6.4.32 of Freund's book for the basis of Zarrinderakht et al's (2022) eq (18)). Nonetheless, K is a function of the dimensional velocity $\dot{d}$, so $K'(0)$ has units of one over velocity, ensuring that the right-hand sides of equation (4) have units of velocity. $K'(0)$ scales approximately as $1/(2c_p) - 1/c_R$, where c_p is the primary wave velocity and c_R the Rayleigh wave velocity. We have changed the text around equation (4) to say

As in Zarrinderakht et al (2022), we assume that each crack propagates at a rate related to how much the stress intensity factor exceeds fracture toughness K_{Ic} by

$$\dot{d}_{\mathrm{b}} = \max\left(-\frac{K_{\mathrm{Ib}} - K_{\mathrm{Ic}}}{K_{\mathrm{Ic}}|K'(0)|}, 0\right), \qquad \dot{d}_{\mathrm{t}} = \max\left(-\frac{K_{\mathrm{It}} - K_{\mathrm{Ic}}}{K_{\mathrm{Ic}}|K'(0)|}, 0\right),$$

where the overdot indicates differentiation with respect to time, and $|K'(0)|$ is the derivative of Freund's (1990) universal function K (given by equation (6.4.26) in Freund's book), evaluated at zero crack propagation velocity. An approximate form of the universal function is $K(\dot{d}) \approx (1 - \dot{d}/v_{\mathrm{R}})/\sqrt{1 - \dot{d}/v_{\mathrm{p}}}$, with v_{R} and v_{p} being Rayleigh and primary wave velocities, so $-1/K'(0) \approx 2v_{\mathrm{p}}v_{\mathrm{R}}/(2v_{\mathrm{p}} - v_{\mathrm{R}})$. As discussed in **?**, *there are alternative hydrofracture-based models for crack tip propagation that could replace this description. We pursue the latter here due to the qualitative insights it provides.* We return to the limitations of the propagation model in the updated section 4.2,

There are several other limitations in addition to not accounting for the effect of buoyancy on elastic stresses. For a given elastic pre-stress, the linear elastic fracture mechanics problem solved here relies on the same weakly inertial propagation rate prescription due to Freund (1990) that was previously used in Zarrinderakht et al (2022). Since the cracks under consideration are typically fluid-filled, it is likely that dynamic propagation is controlled by the retarding effect of fluid flow in the fractures (Spence and Sharpe, 1985), which require a significantly more complicated hydrofracture model, which is unlikely to permit a comprehensive study of parameter space, or even of fracture evolution for different initial conditions as in figures 2, 3 and 5.

Reviewer comment: Eqn 1: Briefly explain the resistive stress Rxx and give a reference

Response: We have added the following after equation (1):

. . . where g is the acceleration due to gravity, and R_{xx} is related to the far-field velocity field U through $R_{xx} = 4\mu\partial U/\partial x$, μ being ice viscosity (Muszynski and Birchfield, 1987, Morland 1987, MacAyeal and Barcilon, 1988).

Reviewer comment: Line 115: Add "Poisson's ratio" before ν

Response: Changed to

. . . is independent of Poisson's ratio ν (while displacements do depend on ν)

Reviewer comment: Line 117: "written in the form" → "written in the dimensionless form"

Response: Changed as suggested.

Reviewer comment: Line 122: As surface crack propagates the dimensionless water level generally won't stay constant. Please add a justification or acknowledge the model limitation caused by this simplification.

Response: The hydrological assumptions behind this work were discussed in detail in Zarrinderakht et al 2022. There we consider 1) a fixed water table, treating the near-surface of the ice shelf as a porous medium able to support an aquifer and 2) a fixed water volume injected into the crack, one being applicable to warm near-surface conditions and the other to cold surface conditions that still have liquid water available (perhaps seasonally?) We assume the referee would rather use the second version, which is however problematic as explained in Zarrinderakht et al 2022 (in the sense that it is quite possible that fracture propagation in 3D should lead to the formation of localized drainage slots, similar to the instability studied in Touvet et al 2011, see Zarrinderakht et al 2022 for full citation). As for the fixed water table height, assuming that there is abundant near-surface water available in the aquifer, it seems unlikely that the water table would drop appreciably due to the small volume accommodated by a crack in an elastic medium.

To make our assumptions clearer, we have added the following after equation (2)

Any part of the upper surface below that elevation is also subject to a hydrostatically increasing water pressure, with the water level remaining unchanged as surface cracks propagate. Implicit here is the presence of a near-surface aquifer that can supply sufficient water to fill the crack while maintaining that constant water level.

Reviewer comment: Line 123: "are constant during crack propagation" -¿ "are assumed to be constant during crack propagation."

Response: We have reworded this to

Any changes in forcing parameters are assumed to occur much more slowly than cracks propagate, so the dimensionless forcing and geometry parameters τ, η, κ, and W are constant during crack propagation.

Reviewer comment: Line 125: "as t increases" → "as time t increases"

Response: Changed as suggested.

Reviewer comment: Line 134: Have the authors checked that when W further increases the result doesn't change for this aligned surface-basal crack case?

Response: We tried a variety values of W including 20, 15, 12, 10, 8 and found no significant difference to our results. This was also done for the results in Zarrinderakht et al (2022), which employ a single crack in the same rectangular geometry. In that case, Tada et al (2000, full reference in the manuscript) provide an interpolated Green's function that was employed in Lai et al (2000, full reference also in the text) and we checked for agreement with that.

Reviewer comment: Line 141: "maximum ensures not only ensures that cracks cannot shrink" The first "ensure" appear to be a typo

Response: Indeed.

Reviewer comment: Line 142: What variable is non-differentiable against what variable? K against d_b, d_t?

Response: Both K's may well be differentiable with respect to d_{t} and d_{b}. That does not mean the right-hand side of equations (7) is differentiable: this is basically a slightly more complicated manifestation of the fact that $f(x) = x$ is differentiable with respect to x, but $g(x) = \max(x, 0)$ is not.

To make this clearer, we have recorded the section slightly to say:

Second, the dynamical system is non-smooth: the maximum function on the right-hand sides of equations (7) not only ensures that cracks cannot shrink, it generally renders those right-hand sides non-differentiable where $K^*_{\mathrm{Ib}} = \kappa$ or $K^*_{\mathrm{Ib}} = \kappa$, even if K_{Ib} and K_{It} are smooth functions of $(d_{\mathrm{b}}, d_{\mathrm{t}})$ (where the latter seems likely unless a new contact area is formed, or a section of open crack fully disappears at that point, see figure 4a of Zarrinderakht et al (2022)).

In other words, we explicitly state that we are concerned with the differnetiability of the right-hand sides of equations (7), not of the K's, and that we actually expect the K's to be differentiable (which we cannot prove). The referenced figure illustrates the non-differentiability for the case of a single crack.

Reviewer comment: Line 144: "intensity factors is equal to the fracture toughness." -¿ "intensity factors is equal to the fracture toughness (green and yellow lines in figure 2)."

Response: That is not strictly speaking true. The original text did not define all of these coloured curves, but they are in fact the orbits into and out of the green-marked saddle point, and the orbits that delimit the basin of attraction of the red-marked node point. They are *nearly* horizontal and vertical for the most part, but not exactly so, as they would need to be if they did coincide with the marginal nullclines. This is probably most evident for the green orbit that emerges near-vertically from the saddle point, which curves noticeably to the right near the calving boundary.

We have changed the text to clarify what the coloured orbits are. In the figure caption for figure 2, *The coloured dots indicate saddle-type (green, yellow and magenta) and node-type (red and cyan) marginal fixed points as defined in the main text. The green curves are the orbits into and out of the green saddle point, while the red and yellow curves are the orbits that delimit the basin of attraction of the red node point.* and in the main text for the red and yellow curves (note that one of the red curves of the original manuscript has been removed as it played no special role, but the yellow orbits of the inset have been extended and added to the main figure panel

Even though the stable node is not an attractor in the strict sense (there are other equilibria arbitrarily close to the stable node), it does have a finite basin of attraction demarcated by the red and yellow orbits into the stable node. Note that he size of that basin of attraction is easy to overestimate visually due to the finite resolution used in computing the phase portrait. Close to the stable node is a marginal fixed point that is analogous to a saddle in standard phase plane analysis, marked with a yellow dot in the inset. For this marginal fixed point, a single orbit ends at the saddle, while a second orbit connects saddle and node (both shown in yellow in the inset). Below the orbit leading up to the saddle, there are additional orbits starting with lower values of $d_\mathrm{t}(0)/d_\mathrm{b}(0)$ *that terminate at the boundary of a region of steady states as shown in the inset.* and for the green curves *Here, one orbit (marked in green) emerges from the saddle point towards the calving boundary while a second orbit (also marked in green) connects a fourth (cyan) marginal fixed point that is almost on the* d_t*-axis to the green saddle point; this orbit is analogous to a separatrix in a standard phase plane, and divides initial conditions that lead to immediate calving from initial conditions that lead to stable, steady cracks of finite length that leave the ice slab intact.*

Reviewer comment: Figure 7: Is the critical stress to drive calving sensitive to the resolution of your numerical model? If yes. Have the authors checked that the result had already converged with higher resolution?

Response: We tested our results by halving and doubling resolution, and found no noticeable change.

Reviewer comment: line 385: "We anticipate that incorporating the feedback between displacement and fluid pressure at the boundary will lead to additional torques generated by vertical displacements in the far field, suppressing crack growth for very large crack spacings". The effect of vertical elastic deformation on buoyancy at the ice-ocean interface was included in Buck and Lai (2021), although their result corresponds to zero fracture toughness. The ice-ocean restoring buoyancy force can increase the critical stress τ_{crit} for basal crack to reach the sea level.

Response: Indeed; this was a restatement from Zarrinderakht et al 2022. Apologies for having dropped the reference to Buck and Lai (2021), which has been restored.