Referee report for “Revisiting the role of vertical shear in analytic ice shelf models” by Miele et al.

In this work, Miele et al. attempt to clarify any confusion that might have arises from the inclusion (or lack thereof) of vertical shear in the equations describing ice shelf flow, and how it relates to the derivation of an expression for depth averaged along-flow deviatoric stress, $\bar{\tau_{xx}} = \rho g h/4$, which is commonly invoked in the literature. They provide a historical perspective, describing different studies which have derived this relationship and the assumptions underpinning such. In particular, they highlight that complete ignorance of vertical shear stress is incompatible with fully neglecting surface slopes. They go on to describe a construction of the vertical shear stress in an ice shelf, which is offered as a way to determine such stresses in situations where they are required.

I found this paper somewhat tricky to review, not least because I don’t think this is necessarily a ‘scientific paper’ in the conventional sense: the main aim of the paper is to clarify misconceptions that might arise on the construction of models in the past. Furthermore, I do not think that the ‘new’ part of the paper (the construction of the shear stress in S7) is indeed new (see below). I think this paper could be useful, particularly to students or those new to the field, but I am unsure whether it is a research paper, per se.

A note on framing: the authors state that “many authors still interpret vertical shear as absent in contemporary ice shelf analysis” and then list many mentions of similar language. As far as I see it, these mentioned authors are saying that the vertical shear stress term is not included at leading order, i.e. that $\partial u_x/\partial z = 0$, which is certainly true to leading order in the aspect ratio (in fact, to order (aspect ratio)^2 — see below written notes, particularly equation 15 therein). The distinction between neglected (in an asymptotic sense) and ignored (i.e. removed from the equations completely) is clear to these mentioned authors, I am sure. However, I am not so sure that this distinction is clear to students and, possibly, those unfamiliar with asymptotic analysis, and thus therein lies the niche of this paper.

More on the asymptotic analysis: this case has been described previously in rigorous detail by Schoof and Hindmarsh (10.1093/qjmam/hbp025, see their “S3.4: Fast Sliding (ii)”). The paper of Schoof and Hindmarsh is fairly intense; below, I have expressed their work in the notation of the present work. In particular, they show that $\partial u_x/\partial z = 0$ at leading order and, although they do not derive it explicitly, it is only a small step from their analysis to the linear stress term of S7.1 of the present paper. Importantly, they do not assume that $\partial \tau_{xx}/\partial z = 0$ (as is assumed by the present paper), but rather show that it emerges at leading order from the Euler equations, i.e. the assumption made in S7 is not necessary. I believe the present paper could be useful in translating this into more digestible language, but the authors should be clear this is not original.

A further point: this analysis shows that $\partial u_x/\partial z = 0$, as assumed by the Thomas model. The authors then go on to show that this leads to a contradiction; however, in the formal asymptotic framework, this is not a contradiction: the terms in their equation 29 are lower order and would be balanced by lower order corrections in the stresses.

Finally, I would say this this paper was quite difficult to read. I offer several suggestions to improve the readability of this paper: (1) many equations are referenced by number a long way from where they are expressed in the text. I would suggest giving them names to prevent having to flick back and forth (e.g. the x momentum equation (6) shows…), (2) I wonder whether it would be clearer to simply explain in words (assuming you do not want to include the rigorous analysis) the different assumptions, and then add derivations in appendices, (3) a table with different models, their assumptions, their expression for deviatoric stress, etc would help the reader to distinguish the models.

$$p + \tau_{xx} = (h - z) + O(\varepsilon^2) \qquad (1)$$

$$\left[\varepsilon = \frac{[z]}{[x]} \text{ aspect ratio} \right]$$

$$2\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} - \frac{\partial h}{\partial x} = O(\varepsilon^2) \qquad (2)$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0 \qquad (3)$$

$$\frac{\partial u_x}{\partial x} = \left(\tau_{xx}^2 + 2\varepsilon^2 \tau_{xz} \right)^{\frac{n-1}{2}} \tau_{xx} \qquad (4)$$

$$\frac{\partial u_x}{\partial z} = 2\varepsilon^2 \left(\tau_{xx} + \varepsilon^2 \tau_{xz} \right)^{\frac{n-1}{2}} \tau_{xz} + O(\varepsilon^2) \qquad (5)$$

Pose asymptotic expansion:

$$p = p_0 + \varepsilon^2 p_2 + \dots \qquad (6)$$

$$\tau_{xx} = \tau_{xx}^0 + \varepsilon^2 \tau_{xx}^2 + \dots \qquad (7)$$

$$\tau_{xz} = \tau_{xz}^0 + \varepsilon^2 \tau_{xz}^2 + \dots \qquad (8)$$

$$u_x = u_x^0 + \varepsilon^2 u_x^2 + \dots \qquad (9)$$

$$u_z = u_z^0 + \varepsilon^2 u_z^2 + \dots \qquad (10)$$

inserting (6) - (10) into (1)-(5) and retaining only leading order terms, we obtain:

$$p_0 + \tau_{xx}^0 = h - z \qquad (11)$$

$$2\frac{\partial \tau_{xx}^0}{\partial x} + \frac{\partial \tau_{xz}^0}{\partial z} = \frac{\partial h}{\partial x} \qquad (12)$$

$$\frac{\partial u_x^0}{\partial x} + \frac{\partial u_z^0}{\partial z} = 0 \qquad (13)$$

$$\frac{\partial u_x^0}{\partial x} = \left(\tau_{xx}^0 \right)^{n-1} \cdot \tau_{xx}^0 \qquad (14)$$

$$\frac{\partial u_x^0}{\partial z} = 0 \qquad (15)$$

(15) implies that $u_x^0 = u_x^0(x)$ ie the horizontal velocity is independent of depth to second order in the aspect ratio!

Then $\frac{\partial u_x^0}{\partial x}$ is independent of z, so, by (14), so is τ_{xx}^0, ie τ_{xx} is independent of z, to leading order in ε^2.

Now we're at the stage reached by Micle et al, after they assume τ_{xx} independent of z.

(Aside: I think the linearity of τ_{xz}^0 can more easily be seen by then taking $\frac{\partial}{\partial z}$ of (12):

$$\underbrace{2\frac{\partial^2 \tau_{xx}^0}{\partial x \partial z}}_{=0 \text{ because } \frac{\partial \tau_{xx}^0}{\partial z} = 0} + \frac{\partial^2 \tau_{xz}^0}{\partial z^2} = \underbrace{\frac{\partial^2 h}{\partial x \partial z}}_{=0 \text{ because } h = h(x)}$$

ie $\frac{\partial^2 \tau_{xz}^0}{\partial z^2} = 0$, (*) so τ_{xz} is linear in z.

Then the solution to (*) satisfying BCs is $\tau_{xz}^0 = \frac{2z}{h} \tau_{xx}^0 \frac{\partial h}{\partial x}$, where τ_{xx}^0 must be found by solving other equations...)

see eqn (3.72) in Schoof + Hindmarsh (2010).