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Rain process models and convergence to point processes

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Abstract. A variety of stochastic models have been used to describe time series of precipitation or rainfall. Since many of these stochastic models are simplistic, it is desirable to develop connections between the stochastic models and the underlying physics of rain. Here, convergence results are presented for such a connection between two stochastic models: (i) a stochastic moisture process as a physics-based description of atmospheric moisture evolution, and (ii) a point process for rainfall time series as spike trains. The moisture process has dynamics that switch after the moisture hits a threshold, which represents the onset of rainfall and thereby gives rise to an associated rainfall process. This rainfall process is characterized by its random holding times for dry and wet periods. On average, the holding times for the wet periods are much shorter than the dry, and, in the limit of short wet periods, the rainfall process converges to a point process that is a spike train. Also, in the limit, the underlying moisture process becomes a threshold model with a teleporting boundary condition. To establish these limits and connections, formal asymptotic convergence is shown using the Fokker-Planck equation, which provides some intuitive understanding. Also, rigorous convergence is proved in mean-square with respect to continuous functions, of the moisture process, and convergence in mean-square with respect to generalized functions, of the rain process.

1 Introduction

Time series of precipitation or rainfall display highly irregular behavior, as illustrated in Fig. 1, and many valuable models have been based on stochastic processes. A variety of different stochastic models have been used, including renewal processes, Markov chains, Poisson processes, and point processes (Green, 1964; Katz, 1977; Richardson, 1981; Smith and Karr, 1983; Foufoula-Georgiou and Lettenmaier, 1987; Rodriguez-Iturbe et al., 1988; Cowpertwait et al., 1996; Wilks and Wilby, 1999). The many applications of these models include weather forecasting, stochastic weather generation, climate impact assessment, climate model downscaling, hydrological modeling, ecological modeling, and agricultural modeling.

Commonly, stochastic models for rainfall are empirical—i.e., based mainly on fitting the model behavior to match observational rainfall data—rather than based mainly on the underlying physical laws. Nevertheless, it is desirable to relate the stochastic models to physical principles, to the extent possible. Here, we investigate such a relation.

In particular, the goal of the present paper is to prove a connection between (i) a point-process description of rainfall time series and (ii) a physics-based model for the stochastic evolution of moisture. At first glance, the point-process model appears to be somewhat disconnected from basic physical laws based on mass, momentum, and energy. However, the point-process

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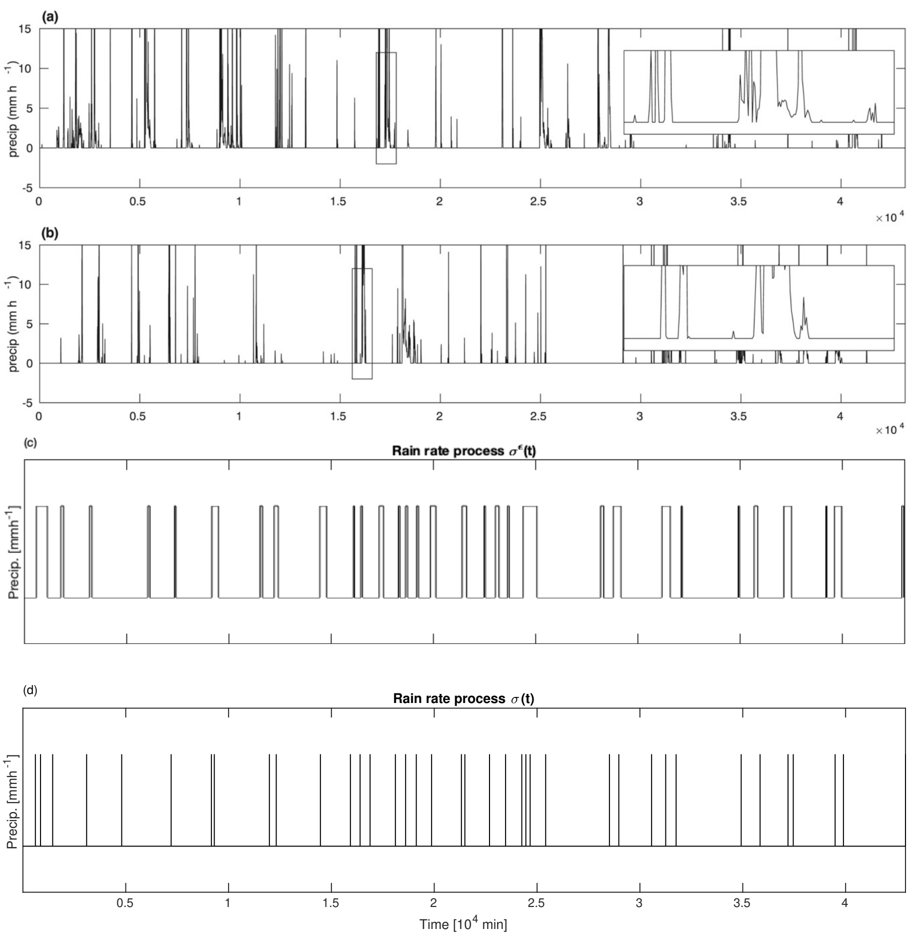

Figure 1. Sample precipitation time series from observations at (a) Manus Island and (b) Nauru Island reproduced from Fig 3. of Abbott et al. (2016) with permission from the authors. The latter two panels are stochastic model simulations of (c) the rain rate process $\sigma^{\epsilon}(t)$ with finite rain rate r and (d) $\sigma(t)$ as the point process.

model can be seen to arise from the underlying evolution of moisture (which is the mass, or mass mixing ratio, of water vapor in the air) (Abbott et al., 2016). Here, this connection is demonstrated via formal asymptotics on the Fokker–Planck equation, and proved rigorously in the mean-square sense.

To be more specific, a point process model of rainfall can be viewed as a spike train, as in Fig. 1d, where a rainfall event is an instantaneous spike. The point process could be defined and characterized by the random waiting time, τ^d, of the duration of the “dry spell” in between rain events. As an empirical model of rainfall, one could estimate the probability density function (pdf) of τ^d based on observational data (Peters et al., 2010; Deluca and Corral, 2014). For such an empirical approach, one could use data of rainfall time series alone, without appealing to any physical laws or any other type of observational data (humidity, wind speed, etc.). Similarly, beyond point processes, one could use a renewal process as a model of rainfall time

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series, as in Fig. 1c, by introducing a finite (and possibly random) time τ^r for the duration of the rain event. Again, as in the case of a point process, one could use a renewal process as an empirical model, based on data of rainfall time series alone, without appealing to any physical laws or any other type of observational data. However, it would be desirable to show that the point process and renewal process models can also arise from more physically based underpinnings.

Here, as mentioned above, a point process model of precipitation will be linked to the evolution of moisture, to provide a more physically based foundation of the point process model. The moisture model used here is a continuous-time stochastic process for $q(t)$, which represents the amount of water vapor in a column of the atmosphere at time t (Stechmann and Neelin, 2011, 2014; Hottovy and Stechmann, 2015b; Abbott et al., 2016; Neelin et al., 2017). The $q(t)$ process is governed by the stochastic differential equations (SDEs)

$$dq(t) = \begin{cases} m\,dt + D_0\,dW_t & \text{for } \sigma(t) = 0 \\ -r\,dt + D_1\,dW_t & \text{for } \sigma(t) = 1 \end{cases}, \quad q(0) = 0, \quad \sigma(0) = 0, \tag{1}$$

where m and r are the moistening and rain rates respectively, and D_0 and D_1 are the fluctuations of moisture during the respective states. The quantity $\sigma(t)$ is an indicator function for rain, and the dynamics of $\sigma(t)$ switch from 0 to 1 when $q(t)$ reaches a fixed threshold $b > 0$. For instance, supposing that $(q(0), \sigma(0)) = (0, 0)$, then $\sigma(t) = 0$ until the time $t_1 = \inf\{t \geq 0 : q(t) = b\}$, at which time the value of σ switches to $\sigma(t) = 1$. Then $\sigma(t)$ switches back to zero at a later time when $q(t)$ reaches a lower threshold, $q(t) = 0$. Figure 2a,b shows a realization of the processes $q(t)$ and $\sigma(t)$. The process $\sigma(t)$ can be viewed as a renewal process, with random durations τ^d and τ^r of dry spells and rain events, respectively, although $\sigma(t)$ is not just a stand-alone renewal process, since it arises from the underlying dynamics of moisture q.

The threshold behavior of (1) is a fundamental feature of the moisture–rainfall relationship that is seen in nature (Peters and Neelin, 2006; Deluca et al., 2015), and it is a basic aspect of many more complex moisture models and convective parameterizations as well (Lin and Neelin, 2000; Frierson et al., 2004; Khouider and Majda, 2005; Khouider et al., 2010; Hottovy and Stechmann, 2015a; Stechmann and Hottovy, 2016; Ahmed and Neelin, 2019; Mueller and Stechmann, 2020; Huang et al., 2022). Sometimes the threshold is also called a trigger (Hernandez-Duenas et al., 2019). The threshold can be viewed as the onset of moist convective instability, and the moisture q is used as the physical quantity that governs the onset of the instability. In this way, (1) is a physically based model of atmospheric moisture, and, from it, one can obtain a rainfall time series as secondary or auxiliary quantity.

The main result of the paper is to define and show convergence of the threshold model in (1) as $r \to \infty$. For example, on the level of renewal processes, $\tau^r \to 0$ and thus $\sigma(t)$ converges to a process that is zero everywhere and has spikes at infinity after random durations of length τ^d. However, $\sigma(t)$ is right continuous and has left hand limits, where as the spike train is not. Thus the mode of convergence is not clear. For $q(t)$ the limit is also unclear, but will be redefined in a way to show convergence with respect to the topology on continuous functions with the uniform metric. In this study, the limiting processes are defined (in Section 2) and convergence is shown both heuristically (for the Fokker-Planck equation) and rigorously.

Some of the novel aspects of this work are as follows. The limit jump process $q(t)$ has an associated Fokker-Planck equation that is derived using a matched asymptotic method. The resulting Fokker-Planck equation has a peculiar boundary flux condi-

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tion which defines a “teleporting” boundary condition of $q(t)$. The processes are decoupled into evaporating and precipitating processes. Only after this decoupling can convergence of the evaporation processes be shown rigorously with respect to the uniform metric on the space of continuous functions. Also, the rain process $\sigma(t)$ is shown to converge rigorously with respect to the generalized function space. This proof shows convergence of a renewal process to a delta process. Furthermore, the proof shows what kinds of bounds are needed for the rain event times τ^r in order for integrated convergence to hold.

The convergence results shown here have the potential to impact various other fields. Many fields of study use similar renewal processes to model different types of phenomena (Cox, 1962). The connections to rain models were made above. In addition, there has been work in queuing theory to approximate point processes with renewal processes (e.g. Whitt (1982); Bhat (1994)), and using threshold triggers in financial models (Lejay and Pigato, 2019). Thresholds arise in many applications of piecewise dynamical systems where the threshold marks a change in the dynamics, as in Fillipov dynamics and hybrid switching diffusions (Filippov, 2013; Simpson and Kuske, 2014). The limiting process is similar to a stochastic resetting process studied in Evans and Majumdar (2011); Evans et al. (2020). Here the process stochastically resets to $q = 0$ after a random hitting time τ^d which depends on the process. Another interesting connection is with neuron stochastic integrate and fire models (see Sacerdote and Giraudo (2013) for a review). The moisture process with a finite rain rate is similar to a Wiener Process model of a single Neuron with refractoriness. A similar model was studied in Albano et al. (2008) where the refractory time was constant. Here, the refractory time is random and coincides with the rain duration time τ^r. Thus the work here is applicable to understanding the differences in using a model without refractoriness versus a model with a short, possible random, refractory time.

The structure of the paper is as follows. The processes for moisture and rain are defined in Section 2. The modes of convergence are discussed in Section 3. The heuristic convergence with the Fokker-Planck equation is shown in Section 3.1. Rigorous convergence of the moistening process E^ϵ to E is shown with respect to L^2 in Section 3.2 and the rain process σ^ϵ is shown to converge to the sum of delta distributions σ with respect to generalized functions in Section 3.3. The results are summarized in Section 4.

2 Model Description

In this section the moisture and precipitation processes are defined. First the underlying moisture process of the renewal rain process is defined. The processes are defined with a small parameter ϵ with the limit as $\epsilon \to 0$ in mind.

The moisture process $q^\epsilon(t) \in \mathbb{R}$ is defined as the solution to the stochastic differential equation (SDE),

$$dq^\epsilon(t) = \begin{cases} m\,dt + D_0\,dW_t & \text{for } \sigma^\epsilon(t) = 0 \\ -\frac{r}{\epsilon}\,dt + D_1\,dW_t & \text{for } \sigma^\epsilon(t) = 1 \end{cases}, \quad q^\epsilon(0) = 0, \quad \sigma^\epsilon(0) = 0, \tag{2}$$

where m and r/ϵ are the moistening and rain rates, and $0 < D_0 \leq D_1$ are the fluctuations of moisture during the respective states. The rain process, $\sigma^\epsilon(t) \in \{0, r/\epsilon\}$ is defined as follows: since $\sigma^\epsilon(0) = 0$, let $\mathcal{T}_1^\epsilon \equiv \inf\{t > 0 | q^\epsilon(t) = b\}$. Then $\sigma^\epsilon(t) = 0$ for $t \in [0, \mathcal{T}_1^\epsilon)$. Next let $\mathcal{T}_2^\epsilon \equiv \inf\{t > \mathcal{T}_1^\epsilon | q^\epsilon(t) = 0\}$, and $\sigma^\epsilon(t) = r/\epsilon$ for $t \in [\mathcal{T}_1^\epsilon, \mathcal{T}_2^\epsilon)$. This process repeats up to an arbitrary

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final time T. Define the time intervals $\tau_i^{d,\epsilon}$ and $\tau_i^{r,\epsilon}$ as

$$\tau_1^{d,\epsilon} = \mathcal{T}_1^\epsilon - 0, \tag{3a}$$

$$\tau_1^{r,\epsilon} = \mathcal{T}_2^\epsilon - \mathcal{T}_1^\epsilon, \tag{3b}$$

$$\tau_2^{d,\epsilon} = \mathcal{T}_3^\epsilon - \mathcal{T}_2^\epsilon, \tag{3c}$$

$$\tau_2^{r,\epsilon} = \mathcal{T}_4^\epsilon - \mathcal{T}_3^\epsilon, \tag{3d}$$

and so on. These are the duration times for dry and rain events.

The associated processes, as $\epsilon \to 0$, are defined as $q(t)$ and $\sigma(t)$ for the moisture and rain processes. (It would perhaps be appropriate to denote the limiting processes as $q^0(t)$ and $\sigma^0(t)$, to indicate that they arise from $q^\epsilon(t)$ and $\sigma^\epsilon(t)$ in the limit $\epsilon \to 0$. However, we will drop the superscript 0 from $q^0(t)$ and $\sigma^0(t)$ to ease notation.) The moisture process is the solution to the SDE,

$$dq(t) = m\,dt + D_0\,dW_t, \quad q < b, \quad q(0) = 0, \tag{4}$$

with the unusual boundary condition as follows: Let the usual stopping time be $\mathcal{T}_1 = \inf\{t > 0 | q(t) = b\}$. Then at time $t > \mathcal{T}_1$ the process $q(t)$ jumps or "teleports" to $q = 0$. Thus

$$\lim_{t \to (\mathcal{T}_1)^-} q(t) = b, \quad \lim_{t \to (\mathcal{T}_1)^+} q(t) = 0, \quad q(\mathcal{T}_1) = b. \tag{5}$$

Then the process starts over using the dynamics of (4) until $\mathcal{T}_2 = \inf\{t > \mathcal{T}_1 | q(t) = b\}$, and the process repeats. The time intervals

$$\tau_i^d = \mathcal{T}_{i+1} - \mathcal{T}_i, \tag{6}$$

are the dry event durations. The rain event duration, on the other hand, is not defined for this limiting process, since rain events are instantaneous in the intense-rain-rate limit of $\epsilon \to 0$.

Example time series of the processes are shown in Figure 2. The processes with finite rain rate r/ϵ for $\epsilon > 0$ are shown in panels (a) and (b). Panel (a) is the moisture process $q^\epsilon(t)$ defined in equation (2). The rain rate process is shown in panel (b) and takes the value r/ϵ when $q^\epsilon(t)$ reaches level b for the first time (panel (a) in black) and resets to zero when $q^\epsilon(t)$ reaches zero (panel (a) in gray). This process repeats. The limiting processes are shown in panels (c) and (d). Panel (c) shows the limiting moisture process $q(t)$ defined in equation (4) and panel (d) shows the rain process defined in equation (7). The moisture process is a Brownian motion with positive drift until reaching level b. When $q(t) = b$, the process $\sigma(t)$ takes an infinite value and the moisture process is reset at zero.

From the definition of τ_i^d above, the rain point process $\sigma(t)$ is defined as

$$\sigma(t) = b \sum_{i=1}^{\mathcal{N}(T)} \delta(t - \mathcal{T}_i), \tag{7}$$

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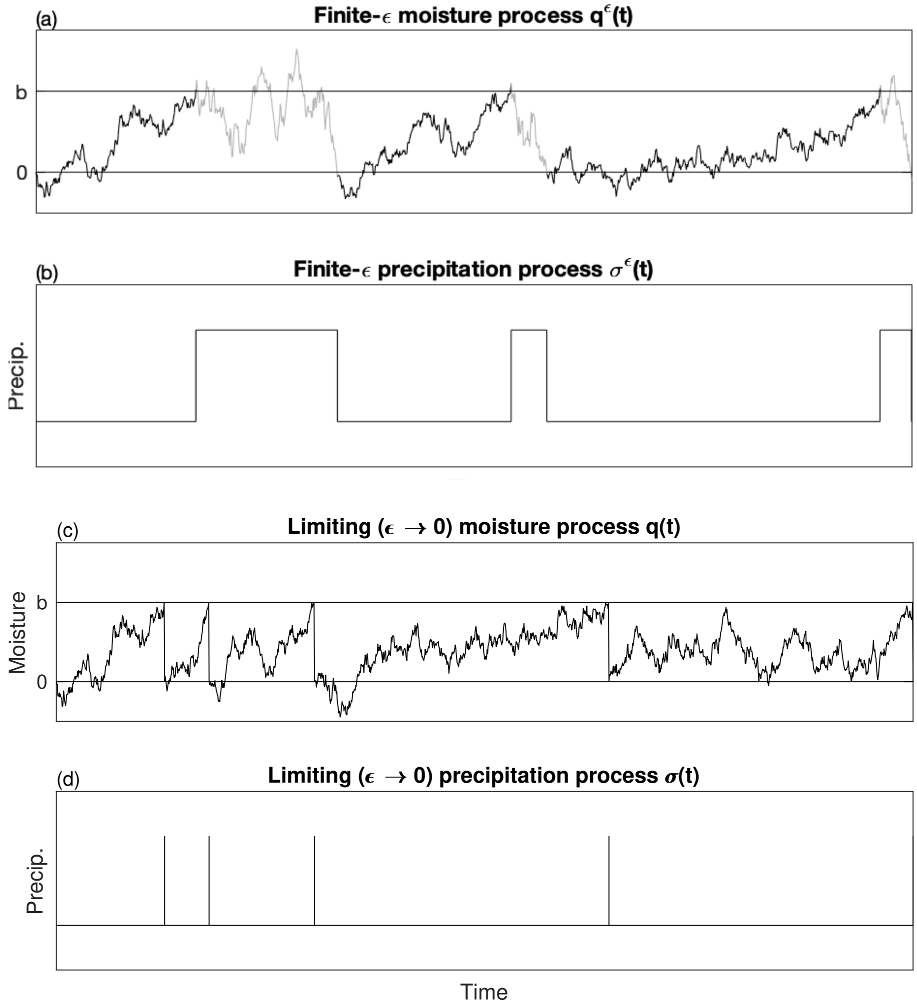

Figure 2. Realizations are plotted of the processes (a) $q^\epsilon(t)$, (b) $\sigma^\epsilon(t)$ for rain rate r/ϵ defined in equation (2) with $\epsilon > 0$ and, on the other hand, the limiting ($\epsilon \to 0$) processes (c) $q(t)$ and (d) $\sigma(t)$ defined in equation (4) and (7) respectively.

where $\mathcal{N}(T)$ is the random variable of the number of times the process $q(t)$ reaches b in time T. The quantity b arises because the moisture process q^ϵ loses moisture at a rate of r/ϵ per time, on average. The moisture process $q(t)$ loses all the moisture built up (which is an amount b) instantaneously.

130 Note that $q^\epsilon(t)$ has continuous paths while $q(t)$ has jump discontinuities. Thus any mode of convergence between q^ϵ and q with an associated metric (e.g. uniform or Skorohod) will fail (Kelley, 2017). Nevertheless, there is another way to define both q^ϵ and q in which convergence with respect to L^2 with the uniform metric on the space of continuous functions ($\mathcal{C}[0,T]$) can be shown. To do so, $q^\epsilon(t)$ is decomposed into an evaporating process, $E^\epsilon(t)$, and precipitating process $P^\epsilon(t)$. These processes

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are defined as

$$dE_t^\epsilon = \begin{cases} m\,dt + D_0 dW_t & \text{for } \sigma_t^\epsilon = 0 \\ 0 & \text{for } \sigma_t^\epsilon = 1 \end{cases}, \quad \text{and} \quad dP_t^\epsilon = \begin{cases} 0 & \text{for } \sigma_t^\epsilon = 0 \\ -\frac{r}{\epsilon}\,dt + D_0 dW_t & \text{for } \sigma_t^\epsilon = 1 \end{cases}. \tag{8}$$

Thus the moisture process $q^\epsilon(t)$ is written as

$$q^\epsilon(t) = E^\epsilon(t) + P^\epsilon(t).$$

In the limit, the jumps will be captured in the P^ϵ process. In the following section it will be shown (see Section 3.2) that $E^\epsilon \to E$, where $E(t)$ is defined as the solution to the SDE

$$dE(t) = m\,dt + D_0\,dW_t, \quad E(0) = 0. \tag{9}$$

Furthermore, the spike times of the σ process, which was defined above in (7), could now also be defined in terms of the $E(t)$ process as $\mathcal{T}_i = \inf\{t > 0 | E(t) = ib,\ i \in \mathbb{N}\}$, i.e. the first passage time of Brownian motion with drift to ib.

3 Convergence to a Point Process

In this section convergence is shown both heuristically (e.g. Section 3.1) and rigorously (e.g. Sections 3.2 and 3.3).

Note that the simplest ideas of convergence break down when considering path-wise convergence of q^ϵ to q and σ^ϵ to σ. This is because q^ϵ is a continuous process for all $\epsilon > 0$, whereas q is a process with jumps; and σ^ϵ is left continuous with right-hand limits, whereas σ no longer is left continuous. Thus, there is no topology with associated metric d such that $q^\epsilon \to q$ with respect to d (Kelley, 2017). However, one could try to show that q^ϵ converges in a notion weaker than the Skorohod topology; see Kurtz (1991) for these conditions. Such convergence would happen in a topology which does not have an associated metric (see Jakubowski et al. (1997)). This approach is not pursued here as it is technical and does not give any insight to the model or approximation.

Instead, we pursue convergence in the following senses. The next three subsections prove convergence of the various processes introduced in Section 2. In Section 3.1 the Fokker-Planck equation for q^ϵ is shown to converge (formally) to a Fokker-Planck equation for q. This derivation gives rise to an interesting partial differential equation (PDE) with unusual "teleporting" boundary conditions. In Section 3.2 convergence in paths is shown for E^ϵ to E with respect to the uniform metric for continuous functions on $[0,T]$. In Section 3.3 convergence is shown for σ^ϵ to σ with respect to generalized functions. This norm is necessary because σ is a sum of Dirac delta functions. In addition, this convergence is natural to consider for applications where the errors are analyzed between using σ^ϵ and a point process (σ) in, for example, a climate model or as a model for observational time series.

3.1 Fokker-Planck Equation

In this section, we derive the Fokker-Planck equation of (4) by taking the formal asymptotic limit, as $\epsilon \to 0$, of the Fokker-Planck equation of (2). This mode of convergence provides some intuition for the behavior in the $\epsilon \to 0$ limit.

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The Fokker-Planck equation for (2) (see Hottovy and Stechmann (2015b)) is composed of two densities. These densities are denoted ρ_0 and ρ_1 for the dry state ($\sigma^\epsilon = 0$) and the rain state ($\sigma^\epsilon = 1$), respectively. These densities evolve according to the following Fokker-Planck equations:

$$\partial_t \rho_0 = -m\partial_q \rho_0 + \frac{D_0^2}{2}\partial_q^2 \rho_0 - \delta(q) f_1\Big|_{q=0}, \quad -\infty < q < b,\, t \geq 0, \tag{10}$$

$$\partial_t \rho_1 = \frac{r}{\epsilon}\partial_q \rho_1 + \frac{D_1^2}{2}\partial_q^2 \rho_1 + \delta(q-b) f_0\Big|_{q=b}, \quad 0 < q < \infty,\, t \geq 0, \tag{11}$$

where the fluxes f_i are defined as

$$f_0(q,t) = m\rho_0(q,t) - \frac{D_0^2}{2}\partial_q \rho_0(q,t) \tag{12a}$$

$$f_1(q,t) = -\frac{r}{\epsilon}\rho_1(q,t) - \frac{D_1^2}{2}\partial_q \rho_1(q,t), \tag{12b}$$

and with the following conditions,

$$\rho_0(b,t) = \rho_1(0,t) = 0, \tag{13}$$

$$\int_{-\infty}^{\infty} \rho_0(q,t) + \rho_1(q,t)\, dq = 1, \quad t \geq 0, \tag{14}$$

which are absorbing boundary conditions and the normalization condition, respectively. One interesting property of these Fokker–Planck equations is the appearance of Dirac-delta source terms, which represent transitions between the dry state and rain state. For instance, in (10), a Dirac delta source term arises at $q = 0$, and it represents the transition from the rain state ($\sigma^\epsilon = 1$) to the dry state ($\sigma^\epsilon = 0$) when the (raining) moisture process reaches the lower threshold at $q = 0$. The magnitude of this Dirac delta source term is $-f_1|_{q=0}$, which is the outward flux of ρ_1 at the lower threshold, $q = 0$, as defined from (11) and (12b).

The proposed limit as $\epsilon \to 0$ for the Fokker-Planck equation is

$$\partial_t \rho_0 = -m\partial_q \rho_0 + \frac{D_0^2}{2}\partial_q^2 \rho_0 + f_0|_{q=b}\delta(q), \quad -\infty < q < b,\, t \geq 0, \tag{15}$$

$$\rho_1 = 0. \tag{16}$$

with the following conditions,

$$\rho_0(b,t) = 0 \tag{17}$$

$$\int_{-\infty}^{b} \rho_0(q,t)\, dq = 1. \tag{18}$$

To derive the limiting ($\epsilon \to 0$) Fokker–Planck equation, the analysis follows the procedure of matched asymptotic expansions (see, e.g., Bender and Orszag (2013)). Consider two regions $[0,\epsilon]$ and $[\epsilon,\infty)$. Let $\rho_{1,B}$ be the density in the first region, which

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is a boundary layer region. For this equation, define the rescaled variable $\tilde{q} = \frac{1}{\epsilon} q$. This yields the equation

$$\partial_t \rho_{1,B} = \frac{r}{\epsilon^2} \partial_{\tilde{q}} \rho_{1,B} + \frac{D_1^2}{2\epsilon^2} \partial_{\tilde{q}}^2 \rho_{1,B} \tag{19}$$

Let $\rho_{1,B}$ have the asymptotic expansion of the form,

$$\rho_{1,B} = \rho_{1,B}^0 + \epsilon \rho_{1,B}^1 + O(\epsilon^2).$$

Substituting this expansion into equation (19) yields, at order ϵ^{-2} and order ϵ^{-1}, respectively,

$$O(\epsilon^{-2}):\ 0 = r\partial_{\tilde{q}} \rho_{1,B}^0 + \frac{D_1^2}{2} \partial_q^2 \rho_{1,B}^0, \tag{20a}$$

$$O(\epsilon^{-1}):\ 0 = r\partial_{\tilde{q}} \rho_{1,B}^1 + \frac{D_1^2}{2} \partial_q^2 \rho_{1,B}^1. \tag{20b}$$

By solving the order ϵ^{-2} equation in (20a) and applying the absorbing boundary condition at $\tilde{q} = 0$, one arrives at

$$\rho_{1,B}^0 = C_1(t) \left(1 - \exp\left[-\frac{2r}{D_1^2}\tilde{q}\right]\right). \tag{21}$$

The order ϵ^{-1} equation in (20b) has essentially the same solution as above, and, after applying the absorbing boundary condition, one finds

$$\rho_{1,B}^1 = C_2(t) \left(1 - \exp\left[-\frac{2r}{D_1^2}\tilde{q}\right]\right). \tag{22}$$

Now consider the interval away from the boundary $[O(\epsilon), \infty)$. Let $\rho_{1,A}$ be the density in this region. The equation in this region is

$$\partial_t \rho_{1,A} = \frac{r}{\epsilon} \partial_q \rho_{1,A} + \frac{D_1^2}{2} \partial_q^2 \rho_{1,A} + \delta(q-b) f_0(b,t). \tag{23}$$

Let $\rho_{1,A}$ have the asymptotic expansion

$$\rho_{1,A} = \rho_{1,A}^0 + \epsilon \rho_{1,A}^1 + O(\epsilon^2).$$

Note that the δ term acts on f_0 which is a function of ρ_0. The asymptotic expansion is for ρ_1 only in the $[O(\epsilon), \infty)$ region, and thus the density ρ_0 is an order one term. Substituting the expansion into equation (23) gives the following equations, separated into their orders of ϵ,

$$O(\epsilon^{-1}) : 0 = r\partial_q \rho_{1,A}^0 \tag{24a}$$

$$O(1) : \partial_t \rho_{1,A}^0 = r\partial_q \rho_{1,A}^1 + \frac{D_1^2}{2} \partial_q^2 \rho_{1,A}^0 + \delta(q-b) f_0(b,t). \tag{24b}$$

The order ϵ^{-1} equation in (24a) has the solution

$$\rho_{1,A}^0 = C_3(t). \tag{25}$$

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Note that $\rho_{1,A}$ is a density and thus $\rho^0_{1,A}$ must be integrable on $[O(\epsilon),\infty)$. Thus $C_3(t)=0$ and

$$\rho^0_{1,A}=0. \tag{26}$$

From the order one equation in (24b), by substituting in $\rho^0_{1,A}=0$, we arrive at

$$\rho^1_{1,A}=\begin{cases} C_4(t) & \text{for} \quad O(\epsilon)\le q<b \\ C_4(t)-\frac{1}{r}f_0(b,t), & \text{for} \quad q\ge b \end{cases} \tag{27}$$

210 Note that the constant of integration in each interval of b must be the same. Otherwise, the magnitude of the δ function in (24b) would not be correct. The density $\rho^1_{1,A}$ must be integrable, which implies that

$$C_4(t)=\frac{1}{r}f_0(b,t). \tag{28}$$

It is assumed that the matching between the A and B solutions must occur at an intermediate location or overlapping region. That is, for values of $q=O(\epsilon^{1/2})$,

$$\rho^0_{1,B}(O(\epsilon^{1/2}),t)=\rho^0_{1,A}(O(\epsilon^{1/2}),t)$$

and

$$\rho^1_{1,B}(O(\epsilon^{1/2}),t)=\rho^1_{1,A}(O(\epsilon^{1/2}),t).$$

215 The first equation implies that $C_1(t)=0$ and $\rho^0_{1,B}=\rho^0_{1,A}=0$. In the limit as $\epsilon\to 0$ the second equation yields

$$C_2(t)=\frac{1}{r}f_0(b,t). \tag{29}$$

Thus the densities are

$$\rho^0_1=0 \tag{30}$$

and

220 $$\rho^1_1=\begin{cases} \frac{1}{r}f_0(b,t)\left(1-\exp\left[-\frac{2r}{D_1^2}\frac{q}{\epsilon}\right]\right) & 0\le q\le O(\epsilon) \\ \frac{1}{r}f_0(b,t) & O(\epsilon)\le q\le b \\ 0 & b<q \end{cases}. \tag{31}$$

Note the flux of ρ_1 at $q=0$ is, to leading order, in terms of ρ^1_1,

$$f_1(0,t)=r\rho^1_1(0,t)+\epsilon\frac{D_1^2}{2}\partial_q\rho^1_1(0,t). \tag{32}$$

Using the asymptotic formula for ρ^1_1 yields

$$f_1(0,t)=\frac{D_1^2}{2}\epsilon\left\{\frac{1}{r}f_0(b,t)\left(\frac{2r}{D_1^2\epsilon}\right)\right\}=f_0(b,t). \tag{33}$$

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Consequently, while the rain-state density itself is small (i.e., $\rho_1^0 = 0$), the flux f_1 of the rain state is $O(1)$, and its value $f_1(0,t)$ at the threshold $q = 0$ represents an $O(1)$ flux from the rain state to the dry state.

Thus the Fokker-Planck type equation for $q(t)$ is

$$\partial_t \rho_0 = -m\partial_q \rho_0 + \frac{D_0^2}{2}\partial_q^2 \rho_0 + f_0|_{q=b}\delta(q), \quad -\infty < q < b, t \geq 0, \tag{34}$$

$$\rho_1 = 0. \tag{35}$$

with the following conditions,

$$\rho_0(b) = 0, \tag{36}$$

$$\int_{-\infty}^{b} \rho_0(q,t)\, dq = 1. \tag{37}$$

Notice an interesting property of this Fokker–Planck equation: the absorbing boundary condition at $q = b$ in (36) is actually coupled to a Dirac-delta source at $q = 0$ in the Fokker–Planck equation in (34). In this coupling, the flux $f_0|_{q=b}$ of absorption at the boundary is also equal to the magnitude of the source $f_0|_{q=b}\delta(q)$ which inserts mass at $q = 0$. Therefore, when the process is absorbed at $q = b$, it is re-inserted at $q = 0$, and in this way it represents a teleporting boundary condition.

3.2 Pathwise Convergence

Rigorous mathematical convergence is now considered. For this section and next, a useful lemma is first stated and proved. In essence, the lemma states that, for a finite time interval $[0,T]$, it is (exponentially) unlikely that a large number of rain events will occur.

Lemma 38. *Let $\mathcal{N}^\epsilon(T)$ be the number of rain events for the q^ϵ process defined in* (4)*. The probability that the number of events is N decays exponentially as N tends to infinity, i.e. for $0 < s < \min\{rb/\epsilon D_2^2, mb/D_1^2\}$*

$$P(\mathcal{N}^\epsilon(T) = N) \leq \exp\left\{ sT - \frac{Nmb}{D_1^2}\left(\sqrt{1 + \frac{2D_1^2 s}{m^2}} - 1\right)\right\}.$$

Proof. Note that the process $\mathcal{N}^\epsilon(T)$ is a renewal process. It is defined by the interarrival times,

$$S_n = \tau_n^{d,\epsilon} + \tau_n^{r,\epsilon}, \quad n \geq 1, \tag{39}$$

where $\tau_i^{d,\epsilon}$ ($\tau_i^{r,\epsilon}$) is the duration for the ith dry (rain) event of the σ^ϵ process. Note that S_n is used instead of $\mathcal{T}_n - \mathcal{T}_{n-2}$ to align with common notation of renewal processes. The distributions of τ_i^d and $\tau_i^{d,\epsilon}$ are the same and are independent of ϵ, while $\tau_i^{r,\epsilon}$ depends on ϵ. For the lemma, the quantity of interest is the probability of having N rain events in time T, which is defined as

$$P(\mathcal{N}^\epsilon(T) = N) = P(S_1 + S_2 + \cdots S_N \leq T, \quad S_1 + S_2 + \cdots + S_N + S_{N+1} > T). \tag{40}$$

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The probability on the right hand side is estimated crudely by only considering one of the two events. Note that $S_1, S_2, S_3, ..., S_n$ are IID random variables with $E[S_1] = E[\tau^{d,\epsilon} + \tau^{r,\epsilon}]$, and $\sigma^2 = \text{Var}(S_1) < \infty$, so that

$$P(\mathcal{N}^\epsilon(T) = N) \leq P\left(S_1 + S_2 + \cdots + S_N \leq T\right). \tag{41}$$

The above probability is estimated by using a variant of the Chernoff bound (Hoeffding, 1994). That is,

$$P\left(S_1 + S_2 + \cdots S_N \leq T\right) \leq \exp(sT) \prod_{i=0}^{n} E[e^{-sS_i}], \tag{42}$$

for any $s > 0$, where $E[e^{-sS_i}] = M_{S_i}(s)$ is the moment generating function for the random variable S_i. The moment generating function can be factored due to independence of $\tau_i^{r,\epsilon}$ and $\tau_i^{d,\epsilon}$:

$$M_{S_i}(s) = M_{\tau_i^{r,\epsilon}}(s) M_{\tau_i^{d,\epsilon}}(s). \tag{43}$$

These moment generating functions are computed explicitly from the distributions found in Hottovy and Stechmann (2015b). They are,

$$M_{\tau_i^{r,\epsilon}} = \int_0^\infty e^{-st} \rho_r(t)\, dt = \exp\left\{ \frac{-rb}{\epsilon D_2^2} \left(\sqrt{1 + \frac{2D_2^2 s \epsilon^2}{r^2}} - 1 \right) \right\}, \tag{44}$$

$$M_{\tau_i^{d,\epsilon}} = \int_0^\infty e^{-st} \rho_d(t)\, dt = \exp\left\{ \frac{-mb}{D_1^2} \left(\sqrt{1 + \frac{2D_1^2 s}{m^2}} - 1 \right) \right\}, \tag{45}$$

which are defined for $s < \min\{rb/\epsilon D_2^2, mb/D_1^2\}$. Chernoff's bound then yields

$$\begin{aligned}
P(\mathcal{N}^\epsilon(T) = N) &\leq P\left(S_1 + S_2 + \cdots S_n \leq T\right) && (46)\\
&\leq \exp(sT) \prod_{i=0}^{N} E[e^{-sS_i}] && (47)\\
&= \exp\left\{ sT - \frac{Nrb}{\epsilon D_2^2} \left(\sqrt{1 + \frac{2D_2^2 \epsilon^2 s}{r^2}} - 1 \right) - \frac{Nmb}{D_1^2} \left(\sqrt{1 + \frac{2D_1^2 s}{m^2}} - 1 \right) \right\} && (48)\\
&\leq \exp\left\{ sT - \frac{Nmb}{D_1^2} \left(\sqrt{1 + \frac{2D_1^2 s}{m^2}} - 1 \right) \right\}. && (49)
\end{aligned}$$

□

With this lemma, pathwise convergence is now considered. Recall from the discussion at the beginning of the section that we consider convergence not for q^ϵ but for the evaporating process E^ϵ. Convergence from E^ϵ to E is shown in $L^2(\Omega)$ with respect to the uniform metric on the space of continuous functions $C[0,T]$.

Theorem 50. *Let q_t^ϵ be defined as*

$$q_t^\epsilon = E_t^\epsilon + P_t^\epsilon$$

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where $E_t^\epsilon, P_t^\epsilon$ are solutions to the SDEs in (8). *Furthermore let E_t be defined as the solution to* (9). *Then*

$$\lim_{\epsilon\to 0} E\left[\left(\sup_{0\le t\le T} |E_t^\epsilon - E_t|\right)^2\right] = 0. \tag{51}$$

Proof. To begin, note that the SDEs for E^ϵ and E (see (8)) only differ when $\sigma^\epsilon(t) = 1$. Thus, the solutions to the SDEs give the formula

$$|E^\epsilon(t) - E(t)| = \left| \sum_{i=1}^{\mathcal{N}^\epsilon(T)} \int_{\mathcal{T}_{2i-1}^\epsilon}^{\mathcal{T}_{2i-1}^\epsilon+\tau_i^{r,\epsilon}} m\, dt + \int_{\mathcal{T}_{2i-1}^\epsilon}^{\mathcal{T}_{2i-1}^\epsilon+\tau_i^{r,\epsilon}} D_0\, dW_t \right|, \tag{52}$$

where $\mathcal{N}^\epsilon(T)$ is the number of rain events for $T < \infty$ and $\epsilon > 0$ fixed. Note that interval $[\mathcal{T}_{2i-1}^\epsilon, \mathcal{T}_{2i}^\epsilon]$ has been written as $[\mathcal{T}_{2i-1}^\epsilon, \mathcal{T}_{2i-1}^\epsilon + \tau_i^{r,\epsilon}]$ to emphasize the rain event duration $\tau_i^{r,\epsilon}$. To proceed, the number of rain events is conditioned to be N. Note that $m > 0$ and the stochastic integral is a martingale and Doob's maximal inequality yields,

$$\begin{aligned} &E\left[\left(\sup_{0\le t\le T} |E^\epsilon(t) - E(t)|\right)^2\right] \\ &\quad \le \sum_{N=1}^{\infty} 4E\left[\left| \sum_{i=1}^{N} \int_{\mathcal{T}_{2i-1}^\epsilon}^{\mathcal{T}_{2i-1}^\epsilon+\tau_i^{r,\epsilon}} m\, dt + \int_{\mathcal{T}_{2i-1}^\epsilon}^{\mathcal{T}_{2i-1}^\epsilon+\tau_i^{r,\epsilon}} D_0\, dW_t \right|^2 \,\middle|\, \mathcal{N}^\epsilon(T) = N\right] P(\mathcal{N}^\epsilon(T) = N). \end{aligned} \tag{53}$$

By Lemma 38, the sum above converges due to the fast decay of $P(\mathcal{N}^\epsilon(T) = N)$ as $N \to \infty$. Applying the Cauchy-Schwarz inequality to the sum and the Itô isometry to the stochastic integral yields

$$E\left[\left(\sup_{0\le t\le T} |E^\epsilon(t) - E(t)|\right)^2\right] \le \sum_{N=1}^{\infty} C_N E\left[m^2|\tau^{r,\epsilon}|^2 + D_0^2|\tau^{r,\epsilon}| \,\middle|\, \mathcal{N}^\epsilon(T) = N\right] P(\mathcal{N}^\epsilon(T) = N), \tag{54}$$

where $\tau^{r,\epsilon}$ denotes the general event duration, which has the same distribution as all of the IID $\tau_i^{r,\epsilon}$. This sum converges due to the fast decay of $P(\mathcal{N}^\epsilon(T) = N)$ as shown in Eq.(49).

To finish the proof the following moments of $\tau_i^{r,\epsilon}$ are used. The integrals can be computed exactly using the densities for $\tau_i^{r,\epsilon}$ found in Hottovy and Stechmann (2015b). They are

$$E[\tau_i^{r,\epsilon}] = \frac{b\epsilon}{r}, \quad E[|\tau_i^{r,\epsilon}|^2] = \frac{bD^2\epsilon^3}{r^3} + \frac{b^2\epsilon^2}{r^2}. \tag{55}$$

Thus the limit is

$$\begin{aligned} \lim_{\epsilon\to 0} E\left[\left(\sup_{0\le t\le T} |E^\epsilon(t) - E(t)|\right)^2\right] &\le \lim_{\epsilon\to 0} \sum_{N=1}^{\infty} C_N E\left[m^2|\tau^{r,\epsilon}|^2 + D_0^2|\tau^{r,\epsilon}| \,\middle|\, \mathcal{N}^\epsilon(T) = N\right] P(\mathcal{N}^\epsilon(T) = N), &(56)\\ &\le \sum_{N=1}^{\infty} \lim_{\epsilon\to 0} C_N \left[m^2\left(\frac{bD^2\epsilon^3}{r^3} + \frac{b^2\epsilon^2}{r^2}\right) + D_0^2\left(\frac{b\epsilon}{r}\right)\right] P(\mathcal{N}^\epsilon(T) = N) &(57)\\ &= 0, &(58) \end{aligned}$$

where Tonelli's theorem allows the limit as $\epsilon \to 0$ to exchange with the infinite sum. This completes the proof. □

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3.3 Distributional Convergence

In this subsection $L^2(\Omega)$ convergence of σ^ϵ to σ is shown with respect to a generalized function norm. This norm is considered here due to the nature of the delta function. It is also a natural norm to consider as it is an integrated error. That is, this norm considers the accumulation of errors after running the model for time $T > 0$.

Theorem 59. *Let $\phi : [0,T] \to \mathbb{R}$ be a test function in $C_c^\infty(0,T)$. Let $\sigma^\epsilon(t)$ and $\sigma(t)$ be defined as in* (2) *and* (7)*, respectively. Then*

$$\lim_{\epsilon\to 0} E[|\langle \sigma^\epsilon(t), \phi(t)\rangle - \langle \sigma(t), \phi(t)\rangle|^2] = 0, \tag{60}$$

where

$$\langle f(t), g(t)\rangle = \int_0^T f(t)g(t)\,dt. \tag{61}$$

Proof. To prove the theorem, the expectation is conditioned on the number of events $\mathcal{N}^\epsilon(T)$, as was done in the previous section. Thus the expectation is

$$\begin{aligned} &E[\langle \sigma^\epsilon(t) - \sigma(t), \phi(t)\rangle^2] \\ &= \sum_{N=1}^{\infty} E\left[\left(\sum_{i=1}^{N} \int_{\mathcal{T}_{2i-1}^\epsilon}^{\mathcal{T}_{2i-1}^\epsilon + \tau_i^{r,\epsilon}} \sigma^\epsilon(t)\phi(t)\,dt \; - \int_0^T b\delta(t-\mathcal{T}_i)\phi(t)\,dt \right.\right. \\ &\qquad \left.\left. - \sum_{i=N+1}^{\mathcal{N}(T)} \int_0^T b\delta(t-\mathcal{T}_i)\phi(t)\,dt\right)^2 \middle| \mathcal{N}^\epsilon(T) = N\right] P(\mathcal{N}^\epsilon(T) = N) \end{aligned} \tag{62}$$

where $\mathcal{N}(T)$ is the number of dry events for the $\sigma(t)$ process up to time T. Again, because of the decay of $P(\mathcal{N}^\epsilon(T) = N)$ as $N \to \infty$ given in Lemma 38, the infinite sum converges.

To estimate the quantity in (62), one rain event is considered and the Cauchy-Schwarz bound will be used. Consider the ith rain event,

$$\begin{aligned} &\int_{\mathcal{T}_{2i-1}^\epsilon}^{\mathcal{T}_{2i-1}^\epsilon + \tau_i^{r,\epsilon}} \sigma^\epsilon(t)\phi(t)\,dt - b\phi(\mathcal{T}_i) \\ &= \int_{\mathcal{T}_{2i-1}^\epsilon}^{\mathcal{T}_{2i-1}^\epsilon + \tau_i^{r,\epsilon}} \frac{r}{\epsilon}\phi(t) - \frac{r}{\epsilon}\phi(\mathcal{T}_{2i-1}^\epsilon) + \frac{r}{\epsilon}\phi(\mathcal{T}_{2i-1}^\epsilon)\,dt + b\phi(\mathcal{T}_{2i-1}^\epsilon) - b\phi(\mathcal{T}_{2i-1}^\epsilon) - b\phi(\mathcal{T}_i) \\ &= \int_{\mathcal{T}_{2i-1}^\epsilon}^{\mathcal{T}_{2i-1}^\epsilon + \tau_i^{r,\epsilon}} \frac{r}{\epsilon}\left(\phi(t) - \phi(\mathcal{T}_{2i-1}^\epsilon)\right)\,dt + \left(\frac{r}{\epsilon}\tau_i^{r,\epsilon} - b\right)\phi(\mathcal{T}_{2i-1}^\epsilon) + b(\phi(\mathcal{T}_{2i-1}^\epsilon) - \phi(\mathcal{T}_i)). \end{aligned} \tag{63}$$

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The function $\phi(t)$ is smooth on $[0,T]$ and thus is locally Lipschitz. Let the Lipschitz constant be $K > 0$. Then, along with the triangle inequality,

$$\left| \int_{\mathcal{T}_{2i-1}^\epsilon}^{\mathcal{T}_{2i-1}^\epsilon+\tau_i^{r,\epsilon}} \sigma^\epsilon(t)\phi(t)\,dt - b\phi(\mathcal{T}_i) \right| \tag{64}$$

$$\leq \int_{\mathcal{T}_{2i-1}^\epsilon}^{\mathcal{T}_{2i-1}^\epsilon+\tau_i^{r,\epsilon}} \frac{r}{\epsilon}K\left|t-\mathcal{T}_{2i-1}^\epsilon\right|\,dt + \left|\left(\frac{r}{\epsilon}\tau_i^{r,\epsilon}-b\right)\phi(\mathcal{T}_{2i-1}^\epsilon)\right| + |\phi(\mathcal{T}_{2i-1}^\epsilon)-\phi(\mathcal{T}_i)| \tag{65}$$

315 $$\leq \frac{r}{\epsilon}K|\tau_i^{r,\epsilon}|^2 + \left|\left(\frac{r}{\epsilon}\tau_i^{r,\epsilon}-b\right)\phi(\mathcal{T}_{2i-1}^\epsilon)\right| + |\phi(\mathcal{T}_{2i-1}^\epsilon)-\phi(\mathcal{T}_i)|, \tag{66}$$

where the last inequality results from $t-\mathcal{T}_{2i-1}^\epsilon$ being an increasing function on $[\mathcal{T}_{2i-1}^\epsilon, \mathcal{T}_{2i-1}^\epsilon+\tau_i^{r,\epsilon}]$.

Using the inequality above, along with the Cauchy-Schwarz inequality, the quantity in (62) is bounded by

$$\sum_{N=1}^{\infty} E\left[\left(\sum_{i=1}^{N}\int_{\mathcal{T}_{2i-1}^\epsilon}^{\mathcal{T}_{2i-1}^\epsilon+\tau_i^{r,\epsilon}} \sigma^\epsilon(t)\phi(t)\,dt - \int_0^T b\delta(t-\mathcal{T}_i)\phi(t)\,dt \right.\right.$$
$$\left.\left. - \sum_{i=N+1}^{\mathcal{N}(T)}\int_0^T b\delta(t-\mathcal{T}_i)\phi(t)\,dt\right)^2 \middle| \mathcal{N}^\epsilon(T)=N\right] P(\mathcal{N}^\epsilon(T)=N)$$

320 $$\leq \sum_{N=1}^{\infty}\sum_{i=1}^{N}\left(\left(\frac{r}{\epsilon}\right)^2 K^2 E[|\tau_i^{r,\epsilon}|^4] + E\left[\left|\left(\frac{r}{\epsilon}\tau_i^{r,\epsilon}-b\right)\phi(\mathcal{T}_{2i-1}^\epsilon)\right|^2\right] + E\left[|\phi(\mathcal{T}_{2i-1}^\epsilon)-\phi(\mathcal{T}_i)|^2\right]\right) P(\mathcal{N}^\epsilon(T)=N)$$
$$+ \sum_{N=1}^{\infty} E\left[\left(\sum_{i=N+1}^{\mathcal{N}(T)}\int_0^T b\delta(t-\mathcal{T}_i)\phi(t)\,dt\right)^2\right] P(\mathcal{N}^\epsilon(T)=N), \tag{67}$$

where all expectations are conditional on $\mathcal{N}^\epsilon(T)=N$.

To finish the theorem the following moments of $\tau_i^{r,\epsilon}$ are used

$$E[\tau_i^{r,\epsilon}] = \frac{b\epsilon}{r}, \quad E[|\tau_i^{r,\epsilon}|^2] = \frac{bD^2\epsilon^3}{r^3} + \frac{b^2\epsilon^2}{r^2}, \quad E[|\tau_i^{r,\epsilon}|^4] = \frac{b^4\epsilon^4}{r^4} + 6\frac{b^3D_1^2\epsilon^5}{r^5} + 15\frac{b^2D_1^4\epsilon^6}{r^6} + 15\frac{bD_1^6\epsilon^7}{r^7}. \tag{68}$$

325 Thus the first term in (67) is

$$\left(\frac{r}{\epsilon}\right)^2 E[|\tau_i^{r,\epsilon}|^4] = O(\epsilon^2). \tag{69}$$

The second term in (67) is

$$E\left[\left|\left(\frac{r}{\epsilon}\tau_i^{r,\epsilon}-b\right)\phi(\mathcal{T}_{2i-1}^\epsilon)\right|^2\right] = E\left[\left(\frac{r}{\epsilon}\tau_i^{r,\epsilon}-b\right)^2\right] E\left[\phi(\mathcal{T}_{2i-1}^\epsilon)^2\right] \tag{70}$$

$$= E\left[\left(\left(\frac{r}{\epsilon}\tau_i^{r,\epsilon}\right)^2 - 2b\frac{r}{\epsilon}\tau_i^{r,\epsilon} + b^2\right)\right] E\left[\phi(\mathcal{T}_{2i-1}^\epsilon)^2\right] \tag{71}$$

330 $$= O(\epsilon) \tag{72}$$

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where the expectation turns into a product because $\tau_i^{r,\epsilon}$ and $\mathcal{T}_{2i-1}^{\epsilon}$ are independent. For the third term of (67), the Lipschitz condition is used to write

$$E\left[|\phi(\mathcal{T}_{2i-1}^{\epsilon}) - \phi(\mathcal{T}_i)|^2\right] \leq K^2 E\left[|\mathcal{T}_{2i-1}^{\epsilon} - \mathcal{T}_i|^2\right]. \tag{73}$$

Note that the stopping times can be written in terms of the moistening processes in the following way:

$$\mathcal{T}_{2i-1}^{\epsilon} = \frac{1}{m}\int_0^{\mathcal{T}_{2i-1}^{\epsilon}} m\,dt, \tag{74}$$

$$= \frac{1}{m}\sum_{j=1}^{2i-1}\int_{\mathcal{T}_{j-1}^{\epsilon}}^{\mathcal{T}_j^{\epsilon}} m\,dt \tag{75}$$

$$= \underbrace{\frac{1}{m}\sum_{j=1}^{i}\int_{\mathcal{T}_{2j-2}^{\epsilon}}^{\mathcal{T}_{2j-1}^{\epsilon}} m\,dt}_{\sigma^{\epsilon}=0} + \underbrace{\sum_{j=1}^{i}\int_{\mathcal{T}_{2j-1}^{\epsilon}}^{\mathcal{T}_{2j-1}^{\epsilon}+\tau_i^{r,\epsilon}} 1\,dt}_{\sigma^{\epsilon}=1} \tag{76}$$

$$\begin{aligned} &= \frac{1}{m}\left(\sum_{j=1}^{i}\int_{\mathcal{T}_{2j-2}^{\epsilon}}^{\mathcal{T}_{2j-1}^{\epsilon}} m\,dt + \int_{\mathcal{T}_{2j-2}^{\epsilon}}^{\mathcal{T}_{2j-1}^{\epsilon}} D_0\,dW_t\right) - \int_0^{\mathcal{T}_{2i-1}^{\epsilon}} \frac{D_0}{m}\,dW_t \\ &\quad + \sum_{j=1}^{i}\left(\int_{\mathcal{T}_{2j-1}^{\epsilon}}^{\mathcal{T}_{2j-1}^{\epsilon}+\tau_i^{r,\epsilon}} 1\,dt + \int_{\mathcal{T}_{2j-1}^{\epsilon}}^{\mathcal{T}_{2j-1}^{\epsilon}+\tau_i^{r,\epsilon}} \frac{D_0}{m}\,dW_t\right) \end{aligned} \tag{77}$$

$$= \frac{1}{m}E_{\mathcal{T}_{2i-1}^{\epsilon}}^{\epsilon} + \sum_{j=1}^{i}\tau_j^{r,\epsilon} + \sum_{j=1}^{i}\int_{\mathcal{T}_{2j-1}^{\epsilon}}^{\mathcal{T}_{2j-1}^{\epsilon}+\tau_i^{r,\epsilon}} \frac{D_0}{m}\,dW_t - \int_0^{\mathcal{T}_{2i-1}^{\epsilon}} \frac{D_0}{m}\,dW_t, \tag{78}$$

where $\mathcal{T}_0^{\epsilon} \equiv 0$. Similarly

$$\mathcal{T}_i = \frac{1}{m}\int_0^{\mathcal{T}_i} m\,dt \tag{79}$$

$$= \frac{1}{m}\left(\int_0^{\mathcal{T}_i} m\,dt + \int_0^{\mathcal{T}_i} D_0\,dW_t\right) - \int_0^{\mathcal{T}_i} \frac{D_0}{m}\,dW_t \tag{80}$$

$$= \frac{1}{m}E_{\mathcal{T}_i} - \int_0^{\mathcal{T}_i} \frac{D_0}{m}\,dW_t, \tag{81}$$

where the Wiener process is the same realization as in (78). The definition of the stopping times $\mathcal{T}_{2i-1}$ and $\mathcal{T}_i$ imply

$$E_{\mathcal{T}_{2i-1}^{\epsilon}}^{\epsilon} = E_{\mathcal{T}_i} = bi.$$

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Thus the difference in stopping times is

$$|\mathcal{T}^{\epsilon}_{2i-1}-\mathcal{T}_i|^2 = \left| \frac{1}{m}(E^{\epsilon}_{\mathcal{T}^{\epsilon}_{2i-1}} - E_{\mathcal{T}_i}) - \int\limits_{\mathcal{T}_i}^{\mathcal{T}^{\epsilon}_{2i-1}} \frac{D_0}{m} dW_t + \sum_{j=1}^{i} \tau_j^{r,\epsilon} + \sum_{j=1}^{i} \int\limits_{\mathcal{T}^{\epsilon}_{2j-1}}^{\mathcal{T}^{\epsilon}_{2j-1}+\tau_i^{r,\epsilon}} \frac{D_0}{m}\, dW_t \right|^2 \tag{82}$$

$$\leq \left| \int\limits_{\mathcal{T}_i}^{\mathcal{T}^{\epsilon}_{2i-1}} \frac{D_0}{m} dW_t \right|^2 + \sum_{j=1}^{i} |\tau_j^{r,\epsilon}|^2 + \sum_{j=1}^{i} \left| \int\limits_{\mathcal{T}^{\epsilon}_{2j-1}}^{\mathcal{T}^{\epsilon}_{2j-1}+\tau_i^{r,\epsilon}} \frac{D_0}{m}\, dW_t \right|^2 . \tag{83}$$

where the triangle inequality has been used. Taking the expected value and using the Itô isometry yields

$$E[|\mathcal{T}^{\epsilon}_{2i-1}-\mathcal{T}_i|^2] \leq \frac{D_0^2}{m^2} E[\mathcal{T}^{\epsilon}_{2i-1}-\mathcal{T}_i] + \sum_{j=1}^{i} E[|\tau_j^{r,\epsilon}|^2] + \frac{D_0^2}{m^2}\sum_{j=1}^{i} E[\tau_j^{r,\epsilon}] \tag{84}$$

$$= \frac{D_0^2}{m^2} E\left[\sum_{j=1}^{i} \tau_j^{d,\epsilon} + \tau_j^{r,\epsilon} - \tau_j^{d}\right] + \sum_{j=1}^{i} E[|\tau_j^{r,\epsilon}|^2] + \frac{D_0^2}{m^2}\sum_{j=1}^{i} E[\tau_j^{r,\epsilon}] \tag{85}$$

Note that $\tau_i^{d,\epsilon}$ and τ_i^{d} are IID random variables with the same distribution and thus the expectations cancel. For the remaining terms, the moments of $\tau_j^{r,\epsilon}$ in equation (68) are used to give

$$K^2 E[|\mathcal{T}^{\epsilon}_{2i-1}-\mathcal{T}_i|^2] \leq K^2 (i\frac{D_0^2}{m^2} E[\tau^{r,\epsilon}] + i\frac{D_0^2}{m^2} E[\tau^{r,\epsilon}] + iE[|\tau^{r,\epsilon}|^2]) = O(\epsilon), \tag{86}$$

which completes the consideration of the third term of (67).

For the last "remainder" term in (67), the expectation is conditioned on both $\mathcal{N}(T)$ and $\mathcal{N}^{\epsilon}(T)$. That is,

$$\sum_{N=1}^{\infty} CE\left[\left(\sum_{i=N+1}^{\mathcal{N}(T)} \int_0^T b\delta(t-\mathcal{T}_i)\phi(t)\, dt \right)^2 \middle| \mathcal{N}^{\epsilon}(T) = N \right] P(\mathcal{N}^{\epsilon}(T)=N) \tag{87}$$

$$= CE\left[\left(\sum_{i=\mathcal{N}^{\epsilon}(T)+1}^{\mathcal{N}(T)} \int_0^T b\delta(t-\mathcal{T}_i)\phi(t)\, dt \right)^2 \right] \tag{88}$$

$$= \sum_{N=1}^{\infty}\sum_{M=1}^{\infty} CE\left[\left(\sum_{i=N+1}^{N+M} \int_0^T b\delta(t-\mathcal{T}_i)\phi(t)\, dt \right)^2 \middle| \mathcal{N}^{\epsilon}(T) = N, \mathcal{N}(T)=N+M \right]$$

$$\cdot P(\mathcal{N}^{\epsilon}(T)=N,\, \mathcal{N}(T)=N+M) \tag{89}$$

If $\mathcal{N}^{\epsilon}(T) \geq \mathcal{N}(T)$, then there is no sum and the term is zero. If $\mathcal{N}^{\epsilon}(T) \neq \mathcal{N}(T)$, then the processes E_t^{ϵ} and E_t from Section 3.3 must be at least b units apart. Thus by theorem 50,

$$P(\mathcal{N}^{\epsilon}(T) \neq \mathcal{N}(T)) = P(|E^{\epsilon}(t) - E(t)| > b). \tag{90}$$

Furthermore, convergence in expectation (L^2) implies convergence in probability. Therefore,

$$\lim_{\epsilon \to 0} P(|E^{\epsilon}(t) - E(t)| > b) = 0. \tag{91}$$

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Putting this together with the above estimate yields

$$\lim_{\epsilon\to 0}\sum_{N=1}^{\infty} CE\left[\left(\sum_{i=N+1}^{\mathcal{N}(T)}\int_0^T b\delta(t-\mathcal{T}_i)\phi(t)\,dt\right)^2 \middle| \mathcal{N}^\epsilon(T)=N\right] P(\mathcal{N}^\epsilon(T)=N) \tag{92}$$

$$\leq \lim_{\epsilon\to 0}\sum_{N=1}^{\infty}\sum_{M=1}^{\infty} CE\left[\left(\sum_{i=N+1}^{N+M}\int_0^T b\delta(t-\mathcal{T}_i))\phi(t)\,dt\right)^2 \middle| \mathcal{N}^\epsilon(T)=N,\mathcal{N}(T)=N+M\right] P(|E^\epsilon(t)-E(t)|>b) \tag{93}$$

$$=0, \tag{94}$$

by using Tonelli's theorem to exchange the sums and limit. Thus all of the terms in (67) have been shown to converge to 0 as $\epsilon\to 0$, so that, returning to (62) and taking the limit, we have

$$\lim_{\epsilon\to 0} E[\langle\sigma^\epsilon(t)-\sigma(t),\phi(t)\rangle^2]=0 \tag{95}$$

and the proof is completed.

□

4 Conclusions

In this paper, a threshold model for moisture and rain was shown to converge to a point process and related processes, and to converge for various modes of convergence. By demonstrating this type of convergence, the simple ideas of a point-process model of rainfall, which at first may appear to be only an empirical model, can be linked with underlying physical processes and evolution of moisture.

Here convergence for the moisture processes was defined and shown for the Fokker-Planck equation as well as the paths of the processes. Furthermore, the convergence of the rain process were shown in mean square difference with respect to the space of generalized functions.

Using a point process to approximate rainfall allows simplification for computation and exact formulas. For example, the autocorrelation function is known in the case of point processes as shown in Abbott et al. (2016). Furthermore, point processes have been studied extensively in the neural science literature (Sacerdote and Giraudo, 2013) and many statistics have been derived.

The proofs shown here are revealing on their own, and they demonstrate further details of the convergence. The Fokker-Planck derivation in Section 3.1 shows that the density for the moisture in the rain state tends to zero while the flux term remains nonzero, allowing for the "teleporting" boundary condition that arises for the limiting moisture process. For the convergence of paths of moisture shown in Theorem 50, the moisture process must first be decoupled into a moistening and precipitating process. Then the moistening process is shown to converge (Theorem 50) while the precipitating process contains all of the discontinuities. Finally, the proof of convergence of the rain processes in Theorem 59 gives estimates that would be useful for determining the error rates for using the point process approximation.

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