Reply to Reviewers
egusphere-2022-865
"Rain process models and convergence to point processes"

Scott Hottovy and Samuel N. Stechmann

December 30, 2022

We would like to thank the reviewer for their comments, which have improved the manuscript in many ways.

One main theme of the revisions has been to better emphasize the importance of the results for a geophysical audience. To do this, the paper has been significantly restructured to make this aim more clear. We have moved many of the technical arguments to the Appendix. To highlight the usefulness of the approximations, we have added an entirely new section.

Below are detailed replies to the reviewer's comments. The replies are structured as followed:

- Referee comments are in magenta.
- The authors' response is in black.
- The author's changes in manuscript are in red.

1 Reviewer #1

Review of "Rain process models and convergence to point processes" by Scott Hottovy and Samuel N. Stechmann discusses the convergence of two coupled diffusion processes, with different diffusion and drift coefficients, to a point process, when one drift term diverges to infinity.

1. The identification with a rain process is questionable as the humidity in the model takes negative values.

   We have added a sentence clarifying that q is an anomaly, and values with $q < 0$ correspond to negative anomalies. It is an anomaly from a baseline level of moisture of 62 mm in the column, which could be taken as the mean value from observational data. So negative values of model anomaly q correspond with moisture values that are below the mean value.

   Even with this addition, yes, the moisture can potentially reach negative values, i.e. $q < -62$. This is due to the simplified nature of the model. Many boundary conditions could be employed at $q = -62$ to maintain positivity – e.g., absorbing boundary, reflective boundary, a soft wall, etc. – or other modifications to the stochastic model could be considered. However, these conditions would complicate the computations given in the paper. We have added statements to the paper clarifying that this is an idealized model which may reach negative values on very rare occasions.

Line 41: "... as an anomaly from a baseline level, at time t (Stechmann and Neelin, 2011, 2014; Hottovy and Stechmann, 2015b; Abbott et al., 2016; Neelin et al., 2017). For example, the anomaly $q(t) = 0$ corresponds to 62 mm of moisture in the column and $q(t) = b = 3$ mm will be an upper threshold of 65 mm of moisture in the column"

Line 53: The moisture in the column can potentially reach negative values, i.e. $q(t) < -62$, on very rare occasions. This effect is due to the idealized nature of the model.

A few of my question are:

2. In line 45 the authors state that "D_0 and D_1 are the fluctuations"; no, they are the diffusion coefficients. Are these coefficients time dependent ?

Yes you are correct. They are constant values. We have corrected the text correspondingly.

Line 46: "...and D_0 and D_1 are the constant diffusion coefficients which capture the fluctuations of moisture during the respective states."

3. I do not understand the boundary conditions (13), if the densities are zero the process will never reach this states and there will never be a jump ? (shouldn't the derivative vanish rather than the value?)

We have added a clarifying discussion to the text. The boundary condition implies the probability of a certain moisture level q and state σ. The absorbing boundary states that that the combination $q = b$, $\sigma = 0$ (or $q = 0$, $\sigma = 1$) does not occur. Indeed it does not because once $q = b$ is achieved, the state changes instantaneously to $\sigma = 1$.

(If the boundary condition were instead defined as a vanishing derivative, as you had wondered about, then the boundary condition would be akin to a reflecting boundary condition.)

Line 181: This implies that once the particle reaches $q = b$ (or $q = 0$) the particle is removed and added to the state $\sigma = 1$ (or $\sigma = 0$) [Gardiner 2004]. Thus the particle reaches $q = b$ but can not be found there $\rho_0(b, t) = 0$.

4. The pdf should converge to a stationary state, a time independent pdf? I think that it can be obtained analytically. This important point is not discussed and I do not understand why the time dependence of the pdf is kept in the calculations of the pdf?

Yes you are correct. The pdf converges to a stationary time independent pdf which can be found analytically. We have added a new section in which a subsection solves the stationary Fokker-Planck and plots its solution.

4.1 Stationary Fokker-Planck Equation

Here the analytical solutions to the stationary Fokker-Planck equation are given. The stationary Fokker-Planck equation for the process q^ϵ is

$$0 = -m\partial_q\rho_0^\infty + \frac{D_0^2}{2}\partial_q^2\rho_0^\infty - \delta(q)f_1, \quad -\infty < q < b, \tag{1}$$

$$0 = \frac{r}{\epsilon}\partial_q\rho_1^\infty + \frac{D_1^2}{2}\partial_q^2\rho_1^\infty + \delta(q-b)f_0, \quad 0 < q < \infty, \tag{2}$$

with the conditions

$$\rho_0(b,t) = \rho_1(0,t) = 0, \tag{3}$$

$$\int_{-\infty}^{\infty} \rho_0(q,t) + \rho_1(q,t)\, dq = 1, \quad t \geq 0, \tag{4}$$

The analytical solutions are found in Hottovy and Stechmann (2015a) and are reproduced here. They are

$$\rho_1^\infty(q) = \frac{1}{b}\frac{m}{r/\epsilon + m}\left\{1 - \exp\left(-\frac{2r}{\epsilon D_1^2}q\right)\right\}, \quad \text{for } 0 \leq q \leq b \tag{5}$$

$$\rho_1^\infty(q) = \frac{1}{b}\frac{m}{r/\epsilon + m}\left\{1 - \exp\left(-\frac{2r}{\epsilon D_1^2}b\right)\right\}\exp\left(-\frac{2r}{\epsilon D_1^2}(q-b)\right), \quad \text{for } 0 \leq q \leq b, \tag{6}$$

and similarly for ρ_0^∞. The densities are plotted in Figure 2. The black curve is the ρ_0^∞ density for one value of $\epsilon = 1$. For the other masses ρ_0^∞ changes very little (order ϵ) and are not shown. The density ρ_1^∞ is plotted in grey for various values of ϵ. The density is scaled by ϵ^{-1}. The dashed gray line is the limiting shape. Thus as $\epsilon \to 0$ the density is a uniform distribution on the interval $[0, b]$ that tends to zero. Note that the absorbing boundary conditions from (14) are satisfied ($\rho_0^\infty(b) = 0$ and $\rho_1^\infty(0) = 0$). However their derivatives (the fluxes) are non-zero. Thus the Dirac delta terms in the Fokker-Planck equation are non-zero.

Additionally, we added a statement on why the time-dependence is kept in the calculations.

Line 214: "The analysis is shown for the full time-dependent Fokker-Planck equation to show that the transient solutions to these equations also converge as $\epsilon \to 0$."

5. There are many points in the paper which I do not understand and the authors should consider sending the paper to a mathematical journal specialized in stochastic processes.

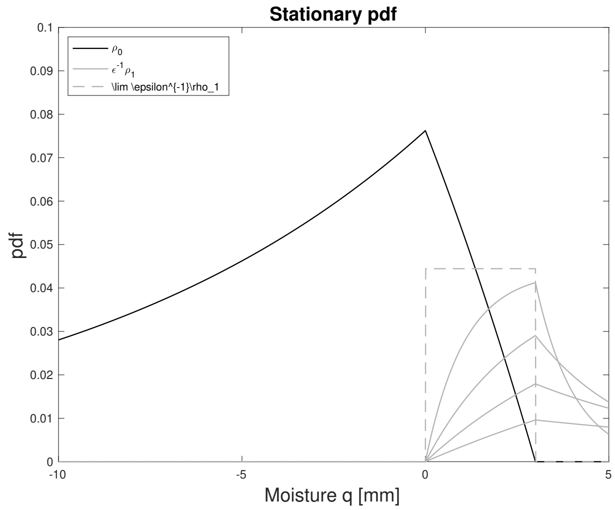

Figure 1: Stationary probability density functions (pdf) for the ρ_0^∞ density (black line) and $\epsilon\rho_1^\infty$ density for values of $\epsilon = 1, 0.5, 0.25$,and 0.1 (gray lines). The dashed line is the limit shape of the ρ_1^∞ density scaled by ϵ^{-1}.

A substantial restructuring and rewriting of the manuscript has been undertaken to better emphasize the importance for a geophysical audience. We have moved many of the technical arguments to the Appendix and have added an additional section which highlights the usefulness of such approximations. In this section we derive and find the stationary Fokker-Planck equations, find analytical rain and dry event distributions, show the error scaling for replacing a rain process σ^ϵ by the point process σ, and give possible augmentations to the point process model to address limitations.

Appendix A

Section 4

2 Reviewer #2

The authors propose a pathway on how to link between a somewhat physically based diffusion processes and more common empirical/statistical pointwise processes. Even though the diffusion processes suggested are not necessarily the most widely accepted models for rainfall. The idea is very good as it may help in guiding future modeling strategies in terms of being able to combine physical intuition with data. The main results consists of a a) rigorous proof in the L2 norm, and b) and not so rigorous and rather very shaky asymptotic expansion, of the former type of processes to the latter. I think the paper can potentially become a very good publication worthy for the readers of NPG-EGUSphere but I conquer that the paper maybe better read if it was submitted to a more math/stats oriented paper as statisticians/mathematicians from other disciplines my appreciate as well as. Nonetheless, the following points may need to be addressed before the paper can be published.

The most serious suggestion I can make is that the authors could simply forgo the section of the asymptotic expansion because it is a mere distraction that doesn't add anything to the main result.

We have moved the asymptotic expansion argument to the Appendix.

Appendix A1

I have a hard time making sense of the asymptotic expansion work which seems to have very serious flaws (see specific comments below). However, I trust that the convergence results in Section 3 are correct although I haven't gone through all the details, especially I haven't checked all the references to make sure that results reported in the literature have been applied correctly, namely because it is my specific area. The revised paper may benefit from being reviewed by a theoretician working in the area of statistics/stochastic processes.

Some of them as miscomprehension/bad notations or typos and some are more serious by they are provided in the order as they appear in the paper.

1. Line 30 - 4:(which is the mixing ratio of water vapor in the air), delete "mass, or mass"

   Yes, the corresponding change has been made.

   LINE 26:"(which is the mixing ratio of water vapor in the air)"

2. Line 45+3: Change "when $q(t)$ reaches a lower threshold" by "when all the moisture is depleted".

   Yes, the corresponding change has been made.

   Line 50: "... when the moisture has been depleted"

3. Line 55+2: Onset of moist convective instability has no meaning. Do you mean to say the onset or convection or the threshold for the release of convective instability — an example is "sampling" parcels of air that have enough energy (buoyancy) to overcome the CIN energy barrier, e.g. see Mapes, B.E.: Convective inhibition, subgridscale triggering energy, and "stratiform instability" in a toy tropical wave model. J. Atmos. Sci. 57, 1515–1535 (2000) Majda, A., and B. Khouider, 2002: Stochastic and mesoscopic models for tropical convection. Proc Natl. Acad. Sci. USA, 99, 1123–1128.

   The wording has now been changed.

   Line 59: "The threshold can be viewed as the threshold for the release of moist convective instability, and the moisture q is used as the physical quantity that governs the onset of moist convection."

4. Line 60: The main purpose instead of the main result. A definition is not a result.

   Yes, the corresponding change has been made.

   Line 63: “The main purpose of the paper...”

5. Line 60: convergence to what?

   The sentence has been completed to signify that the threshold model converges to a point process model.

   Line 63: “...to a point process model of rainfall.”

6. Line 61: Spikes at infinity means here: do you mean that $\sigma(t)$ become a Dirac delta distribution, i.e, the spikes in sigma are infinite? Also, this view is not consistent with Eq 1 where the the values of sigma are either 0 or 1 not zero or infinity. This is very confusing as to what exactly all this means!

   We have changed the definition of Equation (1) to have $\sigma(t) \in \{0, r\}$. This makes the equation consistent with the rest of the paper.

   Equation (1):

$$dq(t) = \begin{cases} m\ dt + D_0\ dW_t & \text{for } \sigma(t) = 0 \\ -r\ dt + D_1\ dW_t & \text{for } \sigma(t) = r \end{cases}, \quad q(0) = 0, \quad \sigma(0) = 0, \tag{7}$$

   We have also added a clarifying statement about “spikes at infinity.”

   Line 65: “That is, $\sigma(t)$ converges to a Dirac delta process.”

7. Line 60+2: whereas is one word.

   Yes, the corresponding change has been made

   Line 66: “...whereas...”

8. Equation 2: Eq 2: should the second value of σ^ϵ be r over epsilon?

Yes, you are correct. The equation has been corrected.

Equation (2): "$\sigma^\epsilon(t) = \frac{r}{\epsilon}$"

9. Line 105-1: These are the duration times for dry and rain events, respectively. Add respectively.

The clarification has been added.

Line 112: "...respectively."

10. Line 110: Should the math expression at the end be $t = T_1$ instead of $t > T_1$

Yes you are correct. A slight change of definition has been made to define the process $q(t)$ as cadlag (continuous from the right with left hand limits). Thus $q(\mathcal{T}_1) = 0$.

Line 118: "Then at time $t = \mathcal{T}_1$ the process $q(t)$ jumps or "teleports" to $q = 0$. However, the function is defined as both 0 and b at $\mathcal{T}_1$. For convention, the process is defined as cadlag (continuous from the right with left hand limits), i.e.

$$\lim_{t \to (\mathcal{T}_1)^-} q(t) = b, \quad \lim_{t \to (\mathcal{T}_1)^+} q(t) = 0, \quad q(\mathcal{T}_1) = 0." \tag{8}$$

11. Line 125-1: to zero instead of at zero.

Yes, the corresponding change has been made

Line 133: "...to zero."

12. Figure 2: Some labeling of some sort should be added to panel d to illustrate the fact sigma is singular, that the spikes are infinite.

Labels have been added to figure 2 to show rain rates of σ^ϵ of r/ϵ and ∞ for σ.

Figure (2): Labels added for panels c) and d).

13. Equation 8: $\sigma^\epsilon = 1 \to \sigma^\epsilon = r/\epsilon$; Should one of the D_0's be D_1 instead?

Yes, you are correct. We have made the changes in the manuscript.

Equation (8):

$$dE_t^\epsilon = \begin{cases} m\,dt + D_0 dW_t & \text{for } \sigma_t^\epsilon = 0 \\ 0 & \text{for } \sigma_t^\epsilon = r/\epsilon \end{cases}, \quad \text{and} \quad dP_t^\epsilon = \begin{cases} 0 & \text{for } \sigma_t^\epsilon = 0 \\ -\frac{r}{\epsilon}\,dt + D_1 dW_t & \text{for } \sigma_t^\epsilon = r/\epsilon \end{cases}. \tag{9}$$

14. Line 165+1: Having both the subscript/assignment notations $q = 0$ and $q = b$ and the Dirac deltas ($\delta(q)$ and $\delta(q-b)$ at the end of the ρ_0 and ρ_1 equations in (10) and (11) is an overkill. One of the two should suffice.

The notation has been simplified by eliminating the previous subscripts.

Equations (10)-(11):

$$\partial_t \rho_0 = -\,m\partial_q \rho_0 + \frac{D_0^2}{2}\partial_q^2 \rho_0 - \delta(q) f_1, \quad -\infty < q < b,\ t \geq 0, \tag{10}$$

$$\partial_t \rho_1 = \frac{r}{\epsilon}\partial_q \rho_1 + \frac{D_1^2}{2}\partial_q^2 \rho_1 + \delta(q-b) f_0, \quad 0 < q < \infty,\ t \geq 0, \tag{11}$$

15. Line 175, end of paragraph: I agree with the authors that the equations in (10), (11) and (12) are both unusual and interesting. However, providing a simple reference for their justification is suboptimal. It will be helpful for the readers if some more discussion is provided, especially in terms of why and/or how the singular terms are obtained. Readers who do not have access or do have the time to read that reference should be given enough information to be able accept/trust these equations.

We have added a new paragraph to describe how equations (10-12) arise from another process studied in Hottovy & Stechmann 2015b.

We hope this new paragraph provides some additional explanation of why and/or how the singular terms are obtained. The subsequent paragraph also provides some explanation, in more intuitive terms.

Line 188: "To obtain the equations (10) and (11) a further approximation is used. In Hottovy & Stechmann 2015b, $q(t)$ is modeled similar to equation (2) except the σ process switches states at a random time with rate λ after $q(t)$ reaches the threshold. Call this process $q^\lambda(t)$. The Fokker-Planck type equation for $q^\lambda(t)$ uses terms from the Master Equation (see Gardiner 2004 Section 3.5). The Fokker-Planck in equation (10) and (11) is derived by taking an asymptotic of the equations for q^λ and taking a limit as $\lambda \to \infty$. That is, as the random switching time becomes small."

Also there are a lot of similarities and also discrepancies between (10) versus (11) and (12a) versus (12b). If I am reading the equations correctly, the singular terms introduce coupling between the two distribution at $q = 0$ and at $q = b$. However, at $\rho_1 = 0$ at $q = 0$ and $\rho_0 = 0$ at $q = b$ may render these coupling terms obsolete, especially if their 1st and second derivatives follow suit, which is likely the cases if the distributions are smooth enough!

We have added statements to clarify the issues you have brought up. You are correct that if the distributions are smooth enough these coupling terms are obsolete. However, the distributions are continuous everywhere but are not differentiable at the boundaries $q = 0$ and $q = b$.

Line 183: "Note that the flux terms in (12a) and (12b) contain ρ_0, ρ_1 terms and its derivative. If the derivatives are zero then the absorbing boundary conditions would imply the Dirac delta coupling terms in (10) and (11) are zero. However, this is not the case. The Fokker-Planck equation must be solved on three separate intervals of $(-\infty, 0]$, $[0, b]$ and $[b, \infty)$. This leads to points of non-differentiability for ρ_0 and ρ_1 at $q = 0$ and $q = b$. For an example, see the stationary solutions in Section 4 and Figure 2."

16. Eqn 15-16: It is a bit weird that the finite epsilon Fokker-Plank equations in (10) and (11) are defined for q in $(-\infty, +\infty)$ but the limiting equations in (15) are restricted to $(-\infty, b)$!!? Some explanation/reconsideration is warranted.

We have added explanation for this fact and pointed to the new section to gain some more intuition. Yes you are correct, that q is restricted for the $\epsilon = 0$ case, and the restriction arises physically because the rainrate is so strong that the moisture q moves above the threshold b by only a small $O(\epsilon)$ amount.

Line 204: "This Fokker-Planck equation is much different than the coupled system (10)-(11). For example, the coupled system is defined for all $-\infty < q < \infty$ whereas (15) is only defined for $-\infty < q \leq b$. This is because the boundary $q = b$ become impassable due to the teleporting boundary; or, in physical terms, the rainrate is so strong that the moisture q moves above the threshold b by only a small $O(\epsilon)$ amount that vanishes as $\epsilon \to 0$. The restriction of q can be seen in the stationary densities in Section 4. There, the stationary density for state $\sigma^\epsilon(t) = 1$ decays to zero quickly."

17. Eqns 19-20: I am not sure I understand the goal nor the effectiveness of asymptotic expansion ansatz. Obviously the nondim parameter in Eq. 19 is ϵ^2 but the ansatz stops short at epsilon level. I am not an expert in asymptotic expansion and I give the authors the benefit of the doubt that what they are doing is most likely correct but I think they owe the reader some motivation about this ansatz. Maybe a better expansion should be in terms of even powers of epsilon only. That way the repetition in (20a) and (20b) wont happen! Without mentioning anything about the higher order terms, $O(\epsilon^2)$ there is no guarantee that there isn't secular growth.

We have restructured our arguments for the asymptotic analysis of the Fokker-Planck equation. This proof is now in the Appendix. To start, we added clarification on why an expansion of ϵ is used.

Line 338: The Fokker-Planck equation for the process q^ϵ is

$$\begin{aligned}
\partial_t \rho_0 =& -m\partial_q \rho_0 + \frac{D_0^2}{2}\partial_q^2 \rho_0 - \delta(q) f_1, \quad -\infty < q < b,\ t \geq 0, \\
\partial_t \rho_1 =& \frac{r}{\epsilon}\partial_q \rho_1 + \frac{D_1^2}{2}\partial_q^2 \rho_1 + \delta(q-b) f_0, \quad 0 < q < \infty,\ t \geq 0,
\end{aligned}$$

To derive the limiting ($\epsilon \to 0$) Fokker–Planck equation, the analysis follows the procedure of matched asymptotic expansions (see, e.g., Bender and Orszag (2013)). Consider two regions $[0, \epsilon]$ and $[\epsilon, \infty)$. Let $\rho_{1,B}$ be the density in the first region, which is a boundary layer region and $\rho_{1,A}$ be the density away from this region. Since the Fokker-Planck equations have parameter ϵ, let $\rho_{1,B}$ have the asymptotic expansion of the form,

$$\rho_{1,B} = \rho_{1,B}^0 + \epsilon \rho_{1,B}^1 + O(\epsilon^2),$$

and let $\rho_{1,A}$ have the asymptotic expansion

$$\rho_{1,A} = \rho_{1,A}^0 + \epsilon \rho_{1,A}^1 + O(\epsilon^2).$$

. These expansions are stopped at the order ϵ level. This is due to the higher order terms not having an impact on the limiting equation. In the end, it will be shown that $\rho_{1,A}^0 = \rho_{1,B}^0$. For the region away from the boundary the density is $\rho_{1,A}^1 = \frac{1}{r} f_0(b,t)$. This allows for the teleporting boundary condition of $f_0(b,t)\delta(q)$.

We added clarification on why we do not consider $O(1)$ or higher PDEs for the asymptotic expansion.

Line 349: "Note that the $O(1)$ equation is not written. At this order and higher, there are iterative PDEs written for $\rho_{1,B}^i$ for $i \geq 2$. These terms will converge to zero at a rate $O(\epsilon^i)$ and thus are not considered here."

18. Line 195+1: Why $O(\epsilon)$ and not simply ϵ as it is initially set in the definition of the two regions above?

The typo has been changed in the manuscript.

Line 356: "Now consider the interval away from the boundary $[\epsilon, \infty)$."

19. Lemma 38 (by the way why this is called lemma 38, this is the only lemma every written in the paper, did I miss something? Same applies for the two theorems.) This lemma should be reworded so that the main result is the inequality-upper bound of the probability distribution as provided. The fact that it decays exponentially as N tends to infinity follows immediately –as a consequence.

Thank you for pointing this out. Lemma 38 has been changed to Lemma 1 in the revised version, and the lemma has been reworded.

Line: 221, "Lemma 1: Let $\mathcal{N}^\epsilon(T)$ be the number of rain events for the q^ϵ process defined in (4). Then for $0 < s < \min\{rb/\epsilon D_2^2, mb/D_1^2\}$

$$P(\mathcal{N}^\epsilon(T) = N) \leq \exp\left\{ sT - \frac{Nmb}{D_1^2}\left(\sqrt{1 + \frac{2D_1^2 s}{m^2}} - 1\right)\right\}."$$

20. Line 270: Should the equation be $\sigma^\epsilon = r/epsilon$?

Yes, you are correct. It has been changed in the manuscript.

Line 423: "To begin, note that the SDEs for E^ϵ and E (see (8)) only differ when $\sigma^\epsilon(t) = r/\epsilon$. "

3 Reviewer #3

Review of the paper "Rain process models and convergence to point processes" by Scott Hottovy and Samuel N. Stechmann.

This paper establishes a novel connection between a widely used empirical point process model for rainfall time and stochastic model for moisture evolution. The authors prove that the moisture model converges to a point process for large rain rates. This is done by using formal asymptotic expansion of the Fokker-Planck equation, as well as, rigorous convergence analysis. Although, I am not an expert on the rigorous analysis and because of this cannot verify the corresponding part (Sec. 3.2, 3.3) of the paper, I agree with the authors that the demonstrated connection is very interesting and revealing. However, the authors should be much more specific about the possible applications of their results in the context of constructing physically constrained models of precipitation. For example, is it possible to make statements about the limitations of existing purely empirical point process models?

An entire new section has been added to discuss potential applications and limitations of the model. This is found in Section 4. This includes subsections of "Stationary Fokker-Planck Equation," "Event Duration," and "Average Cloudiness," which address potential applications. Subsection 4.5 "Modifications for the point process model" discusses limitations to the point process models and how to address these limitations.

Section 4:

4 Statistics and Applications

In this section, important statistics and applications of the processes $(q^\epsilon, \sigma^\epsilon)$ and (q, σ) are presented. These statistics show the differences between the processes as well as give motivation for the approximations. These include the stationary Fokker-Planck solution, the rain and dry event distributions, rain fraction, and an application of Theorem 2.

4.1 Stationary Fokker-Planck Equation

Here the analytical solutions to the stationary Fokker-Planck equation are given. The stationary Fokker-Planck equation for the process q^ϵ is

$$0 = -m\partial_q \rho_0^\infty + \frac{D_0^2}{2}\partial_q^2 \rho_0^\infty - \delta(q) f_1, \quad -\infty < q < b, \tag{12}$$

$$0 = \frac{r}{\epsilon}\partial_q \rho_1^\infty + \frac{D_1^2}{2}\partial_q^2 \rho_1^\infty + \delta(q-b) f_0, \quad 0 < q < \infty, \tag{13}$$

with the conditions

$$\rho_0(b,t) = \rho_1(0,t) = 0, \tag{14}$$

$$\int_{-\infty}^{\infty} \rho_0(q,t) + \rho_1(q,t)\, dq = 1, \quad t \geq 0, \tag{15}$$

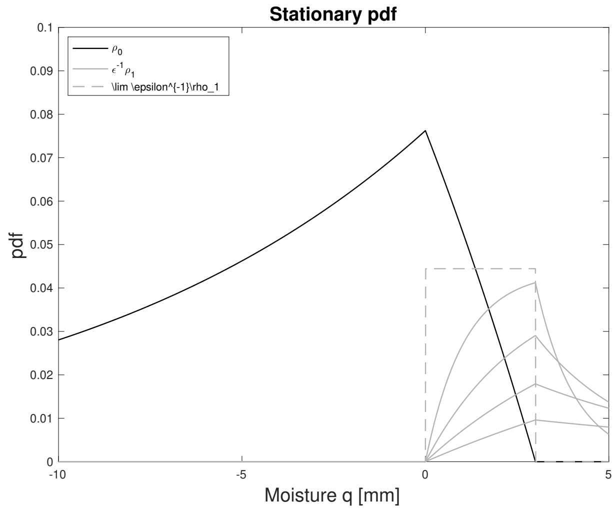

Figure 2: Stationary probability density functions (pdf) for the ρ_0^∞ density (black line) and $\epsilon\rho_1^\infty$ density for values of $\epsilon = 1, 0.5, 0.25$,and 0.1 (gray lines). The dashed line is the limit shape of the ρ_1^∞ density scaled by ϵ^{-1}.

The analytical solutions are found in Hottovy and Stechmann (2015a) and are reproduced here. They are

$$\rho_1^\infty(q) = \frac{1}{b}\frac{m}{r/\epsilon + m}\left\{1 - \exp\left(-\frac{2r}{\epsilon D_1^2}q\right)\right\}, \quad \text{for } 0 \leq q \leq b \tag{16}$$

$$\rho_1^\infty(q) = \frac{1}{b}\frac{m}{r/\epsilon + m}\left\{1 - \exp\left(-\frac{2r}{\epsilon D_1^2}b\right)\right\}\exp\left(-\frac{2r}{\epsilon D_1^2}(q-b)\right), \quad \text{for } 0 \leq q \leq b, \tag{17}$$

and similarly for ρ_0^∞. The densities are plotted in Figure 2. The black curve is the ρ_0^∞ density for one value of $\epsilon = 1$. For the other masses ρ_0^∞ changes very little (order ϵ) and are not shown. The density ρ_1^∞ is plotted in grey for various values of ϵ. The density is scaled by ϵ^{-1}. The dashed gray line is the limiting shape. Thus as $\epsilon \to 0$ the density is a uniform distribution on the interval $[0, b]$ that tends to zero. Note that the absorbing boundary conditions from (14) are satisfied ($\rho_0^\infty(b) = 0$ and $\rho_1^\infty(0) = 0$). However their derivatives (the fluxes) are non-zero. Thus the Dirac delta terms in the Fokker-Planck equation are non-zero.

4.2 Event Duration

Another statistic studied here is the event duration probability density. This density gives information on the probability of a dry/rain event lasting time t minutes. For the process q^ϵ both the dry and rain states are Brownian motions with drift (m for dry and r/ϵ for rain). Thus the event duration densities are the first passage to $q = b$ densities for Brownian motion with drift (Gardiner, 2004, Section 5.5.1). These densities were found in Stechmann and Neelin (2014) and are reproduced here. The event duration density for a rain event is

$$\rho_{1t} = \frac{b}{\sqrt{2\pi D_1^2}}\exp\left\{\frac{rb}{\epsilon D_1^2}\right\}\exp\left\{\frac{-b^2}{2D_1^2 t}\right\}\exp\left\{\frac{-r^2 t}{2\epsilon^2 D_1^2}\right\}t^{-3/2}, \tag{18}$$

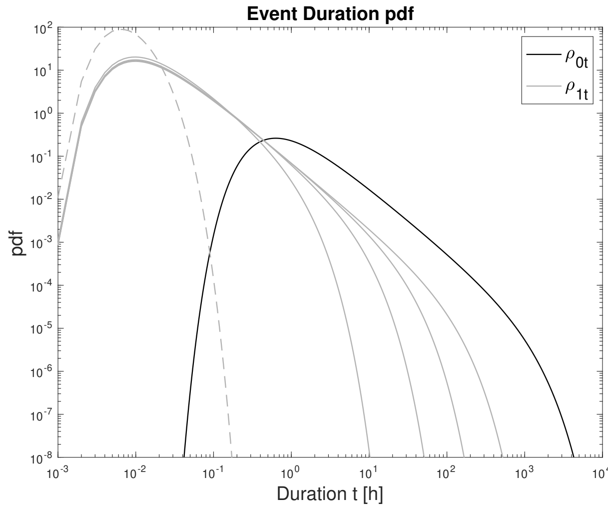

Figure 3: Event duration probability density functions (pdf) for dry events (ρ_{0t} black line) and rain events (ρ_{1t} gray lines) for $\epsilon = 1, 0.5, 0.25,$and 0.1. The dashed line is for $\epsilon = 0.01$.

and similarly for the dry event duration ρ_{0t}. The density for rain events changes with ϵ while the dry event density does not. The rain event duration has cutoffs for short and long times. They are

$$\text{short time cutoff:}\frac{b^2}{2D_1^2} \quad \text{long time cutoff:}\frac{2\epsilon^2 D_1^2}{r^2}.$$

The short time cutoff is independent of ϵ while the long time cutoff tends to zero in $O(\epsilon^2)$. This implies that for smaller ϵ, the extreme events are being cut off quickly. Hence, in the $\epsilon \to 0$ limit, extreme rainfall events do not occur. In order to preserve extreme rainfall events in a point-process model of rainfall, the rain rate amplitude would need to be modeled as a stochastic process, since rain event durations are assumed to be short.

These densities are plotted in Figure 3 on a log-log scale. The dry density ρ_{0t} is the black curve. The rain event duration density ρ_{1t} is plotted in gray for various values of ϵ. The dashed gray line is a density for small $\epsilon = 0.01$.

4.3 Average Cloudiness

The average cloudiness is the fraction of time that the stationary process is in the rain state $\sigma > 0$. It is defined as,

$$E[\sigma/(r/\epsilon)] = \int_0^\infty \rho_1^\infty(q)\, dq$$

where ρ_1^∞ is the solution to the steady state Fokker-Planck equation (13). For the process σ^ϵ it is

$$E[\sigma^\epsilon/(r/\epsilon)] = \frac{1}{m + r/\epsilon}.$$

Futhermore, the variance of the average cloudiness is

$$\mathrm{var}(\sigma^\epsilon/(r/\epsilon)) = \frac{1}{m + r/\epsilon}\left(1 - \frac{1}{m + r/\epsilon}\right) = \frac{m + r/\epsilon - 1}{(m + r/\epsilon)^2}.$$

The average cloudiness is zero when using the point process model σ. However, for σ^ϵ as ϵ tends to zero, the average cloudiness and its variance are $O(\epsilon)$ and nonzero. In some applications, it is important that the cloud fraction is nonzero.

4.4 Total Rainfall

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4.5 Modifications for the point process model

The issue of how to use finite-event-duration model to inform parameter selection is discussed in this subsection. There are two potential points of concerns with the point process model. One is is that the point process model $\sigma(t)$ has rain events of duration zero, whereas the rain process model $\sigma^\epsilon(t)$ has rain events of duration $\tau^{\epsilon,r}$. Thus for time T,

$$T = \sum_{i=1}^{\mathcal{N}(T)} \tau_i^d + \tau_i^{\epsilon,r},$$

where $\mathcal{N}(T)$ is the random number of rain events in T time. For finite $\epsilon > 0$ the value of $\mathcal{N}(T)$ is larger than for the point process model, on average.

There are many potential solutions for the issue of zero event times for the point process model. One example is to modify dry duration pdf to account for small but finite size of rain events. That is, let τ^r be a random variable which models a finite size rain event. For example see equation (18). Define

$$\tau^{d'} = \tau^d + \tau^r,$$

to be a new dry event random variable. This new event distribution would account for the rain event within the dry event. As for the moisture process q, once the threshold of $q = b$ is met, the process then holds at b for a random time of τ^r. Then the process would jump to $q = 0$.

Another potential modification is the definition of rain amount. For the model with finite $\epsilon > 0$, for each event the model rains b amount over a random time $\tau^{r,\epsilon}$. For the point process model, the model rains b amount instantaneously. Here the discrepancy between the two models is captured in Theorem 2 and the example of total rainfall in the example in Subsection 4.4. A possible modification to the model would be to use a finite-event-duration model to help assign a random magnitude to each point-process model event. For example, let b_i be random variables with some distribution from a finite-event-duration model which accounts for random rain amounts. Then the point process model can be modified from Eq. (7) as

$$\sigma(t) = \sum_{i=1}^{\mathcal{N}(T)} b_i \delta(t - \mathcal{T}_i). \tag{19}$$

In the concluding section the authors mention error rates for point processes. Can such estimates be given in the revised paper using some observational data? Similar estimates will demonstrate the potential of the results to the NPG-readers.

We added a subsection to give an example how the estimates from the proof would aid in error estimates. This is done in Subsection 4.4

Subsection 4.4:

4.4 Total Rainfall

For an example of using the results of Theorem 2, consider calculating the total rainfall at a specific grid point of a GCM. Define the rainfall time series at a point as

$$A(t) = \sigma(t)\phi(t)$$

where σ encodes the rainfall rate, and $\phi(t) = 1$ when there is a cloud (or rainfall) and $\phi(t) = 1$ otherwise. One question is: what is the impact on total rainfall in a simulation of time T between the rain processes σ^ϵ and σ?

Theorem 2 states that the integrated difference of the total rainfall tends to zero. That is,

$$\lim_{\epsilon\to 0}\int_0^T |A^\epsilon(t) - A(t)|\, dt = \int_0^T |\sigma^\epsilon(t)\phi(t) - \sigma(t)\phi(t)|\, dt.$$

The proof of the theorem in the Appendix yields more information than that. For example, if it is known that there are N rain events after time T, then estimate (A45), yields

$$\int_0^T |A^\epsilon(t) - A(t)|\, dt \le \sum_{i=1}^{N}\left(\left(\frac{r}{\epsilon}\right)^2 K^2 E[|\tau_i^{r,\epsilon}|^4] + E\left[\left|\left(\frac{r}{\epsilon}\tau_i^{r,\epsilon} - b\right)\phi(\mathcal{T}_{2i-1}^\epsilon)\right|^2\right] + E\left[|\phi(\mathcal{T}_{2i-1}^\epsilon) - \phi(\mathcal{T}_i)|^2\right]\right), \tag{20}$$

where the probability on N events are one and where the final remainder term has been dropped (for clarity of this calculation). Here K is a global bound on the function ϕ. For this example $K = 1$. From the estimates in the proof, the right hand side is bounded by terms which have the first four moments of the event duration time $\tau^{r,\epsilon}$, the global bound K of ϕ, and the number of rain events. Thus, Theorem 2 and its proof give details on the differences for key atmospheric elements when using the rain processes σ^ϵ and σ.

Alternatively, the authors might consider submitting the paper to a more mathematical journal in order to access the mathematical interested readers.

A substantial restructuring and rewriting of the manuscript has been undertaken to better emphasize the importance for a geophysical audience. We have moved many of the technical arguments to the Appendix and have added an additional section which highlights the usefulness of such approximations. In this section we derive and find the stationary Fokker-Planck equations, find analytical rain and dry event distributions, and show the error scaling for replacing a rain process σ^ϵ by the point process σ.

Appendix A

Section 4

Some additional major comments:

1. In eq. 1 the water vapor mass mixing ratio, q, can become negative which is clearly non-physical. Can the approach be modified to account for positive values of q only? If not, this limitation should be discussed in the paper.

We have added a sentence clarifying that q is an anomaly, and values with $q < 0$ correspond to negative anomalies. It is an anomaly from a baseline level of moisture of 62 mm in the column, which could be taken as the mean value from observational data. So negative values of model anomaly q correspond with moisture values that are below the mean value.

Even with this addition, yes, the moisture can potentially reach negative values, i.e. $q < -62$. This is due to the simplified nature of the model. Many boundary conditions could be employed at $q = -62$ to maintain positivity – e.g., absorbing boundary, reflective boundary, a soft wall, etc. – or other modifications to the stochastic model could be considered. However, these conditions would complicate the computations given in the paper. We have added statements to the paper clarifying that this is an idealized model which may reach negative values on very rare occasions.

Line 41: "... as an anomaly from a baseline level, at time t (Stechmann and Neelin, 2011, 2014; Hottovy and Stechmann, 2015b; Abbott et al., 2016; Neelin et al., 2017). For example, the anomaly $q(t) = 0$ corresponds to 62 mm of moisture in the column and $q(t) = b = 3$ mm will be an upper threshold of 65 mm of moisture in the column"

Line 53: The moisture in the column can potentially reach negative values, i.e. $q(t) < -62$, on very rare occasions. This effect is due to the idealized nature of the model.

2. Eq. 2 and line 97. What are the correct values for the rain process $\sigma(t)$: 0, 1 or $\{0, r/\epsilon\}$? Both values can be found at various places in the paper (e.g. equation 8), but since σ converges to a Delta function, r/ϵ should be the correct one.

The correct values are $\sigma^\epsilon(t) = 0$ or r/ϵ. These have been corrected throughout the paper.

Equation (2): "$\sigma^\epsilon(t) = \frac{r}{\epsilon}$"
Equation (8):

$$dE_t^\epsilon = \begin{cases} m\,dt + D_0 dW_t & \text{for } \sigma_t^\epsilon = 0 \\ 0 & \text{for } \sigma_t^\epsilon = r/\epsilon \end{cases}, \quad \text{and} \quad dP_t^\epsilon = \begin{cases} 0 & \text{for } \sigma_t^\epsilon = 0 \\ -\frac{r}{\epsilon}\,dt + D_1 dW_t & \text{for } \sigma_t^\epsilon = r/\epsilon \end{cases}. \tag{21}$$

Line 423: "To begin, note that the SDEs for E^ϵ and E (see (8)) only differ when $\sigma^\epsilon(t) = r/\epsilon$."

3. Fig.2. What value for ϵ was used for Fig.2a and Fig2.b? Axis tick values and labels are missing.

We have added tick values and labels to Figure 2. We have added the values of m, r, and ϵ to the caption. ϵ here was chosen to be 100. Meaning there is a factor of 100 difference between m and r.

Figure 2 caption: "Here $m = r = 0.4$ mm/h with $\epsilon = 0.01$. Thus the effective rain rate is 40 mm/hr"

4. In order to prove convergence large raining rates are assumed; is this large compared to the moistening rate m? Is ϵ defined as m/r ? Can you give an estimate of ϵ from real data? What rain and moistening rates are used in other models, or what are the corresponding dry and rain event duration τ^d and τ^r, respectively?

We added some examples of rain rates seen in observation. In short, ϵ is essentially the ratio of the moistening and rain rates. In terms of the notation used in the paper, since the rain rate is defined as r/ϵ, the value of ϵ is the value which causes $\epsilon m/r$ to be order 1. Estimates from Holloway and Neelin (2010) give $m = 0.4$ mm/hr whereas the rain rate is sometimes in excess of $15mm/hr$. This can be found in their composites in their Figures 7 a and b. So one could define the $O(1)$ constant r to be 1 mm/hr, and then ϵ takes the value of 1/15, in accord with a rain rate r/ϵ of 15 mm/hr.

Line 102: Here the small parameter ϵ is essentially the ratio of the moistening and rain rates (times a constant $O(1)$ factor). In other words, ϵ is the value which makes m/r order 1. In the tropics the rain rate is often seen to have values in excess of 15 mm h^{-1} (e.g. see Figure 1 panels a) and b)). See Figure 7a) in Holloway and Neelin (2010) for an estimate of a typical moistening rate in the tropics. There the moistening rate is roughly 0.4 mm h^{-1}.