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Extension of Ekman (1905) wind-driven transport theory to the β-plane

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Abstract. The seminal, Ekman (1905)'s, f-plane theory of wind driven transport at the ocean surface is extended to the β-plane by substituting the pseudo angular momentum for the zonal velocity in the Lagrangian equation. The addition of the β term implies that equations become nonlinear, which greatly complicates the analysis. Though rotation relates the momentum equations in the zonal and the meridional directions, the transformation to pseudo angular momentum greatly simplifies the longitudinal dynamics, which yields a clear description of the meridional dynamics in terms of a slow drift compounded by fast oscillations, which can then be applied to describe the motion in the zonal direction. Both analytical expressions and numerical calculations underscore the critical role of the equator in determining the trajectories of water columns forced by eastward directed wind stress even when the water columns are far from the equator. Our results demonstrate that the averaged motion in the zonal direction is highly dependent on the meridional oscillations and for some initial conditions can be as large as the meridional mean motion.

1 Introduction

The seminal theory of wind driven transport at the ocean surface was developed about 120 years ago by the Swedish oceanographer Vagn Walfrid Ekman for the highly idealized case of constant Coriolis frequency – the f-plane. The Ekman (1905) theory addresses the downward spiraling horizontal velocity in the ocean's surface and its vertical integral – the transport. Ekman's elegant solution of the problem has become a textbook material in physical oceanography, dynamical meteorology and geophysical fluid dynamics (see e.g. Gill, 1982; Pedlosky, 1987; Vallis, 2017). For uniform wind stress the dynamics on the f-plane consists of two parts: A steady flow to the right/left of the wind direction in the northern/southern hemisphere and inertial oscillations (with frequency f_0 – the constant Coriolis frequency). However, though it is one of the cornerstones of atmosphere and ocean dynamics, the theory was never extended to include increase in the Coriolis frequency with the latitude, known as the β effect, which is the focus of the present study.

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With the wind-driven dynamics on the f-plane fully understood and quantified, the known general differences between the dynamics on the f-plane and β-plane suggest heuristically that the extension of Ekman's transport theory to the β-plane should include the following qualitative elements:

1. An increase/decrease in mean meridional velocity for an eastward/westward directed stress due to the decrease/increase in Coriolis frequency when the water column moves southward/northward.

2. The frequency of oscillation about the mean velocity should decrease/increase (and the oscillation period should increase/decrease) due to the decrease/increase in Coriolis frequency along the trajectory (for an eastward directed stress while the opposite changes for westward directed tress).

3. Since the oscillation's frequency and amplitude are inversely correlated (energy flux is unchanged) a decrease in frequency should lead to an increase in amplitude and vise versa.

4. Since inertial oscillations, that form a perfectly circular motion on the f-plane, drift westward on the β-plane the averaged zonal motion should drift to the west. A heuristic reasoning of the westward drift in terms of the change in the radius of the inertia circle was proposed by Von Arx (1964) and complete quantitative theories of the drift were developed in Ripa (1997) and Paldor (2007).

The numerical solutions of the governing Lagrangian equations (see section 2 below) shown in figure 1 fully confirm the first 3 expectations listed above but contradict the fourth one – for both westward (right panel) and eastward (left panel) stresses, the trajectories drift to the west. From the particular example shown in figure 1 it is unclear whether the eastward transition is a general feature of the wind driven dynamics on the β-plane or a specific occurrence related to the particular choice of initial conditions and/or parameter values.

In addition to resolving the issue of the zonal drift and quantifying the various rates of changes the present study also addresses the following equatorial problem that exists only on the β-plane. This equatorial issue can be described as follows: An eastward directed stress in the northern hemisphere forces a net southward directed mean flow which, on the β-plane, is accompanied by an indefinite decrease in the Coriolis frequency. At some time the wind forced water column must find itself in a latitude where the Coriolis frequency vanishes – the equator. From that point onward the water column is subject to non-rotating dynamics and must move eastward at an accelerated velocity. In the rest of this work we will estimate the time it takes the water column to change its dynamics qualitatively to a non-rotating dynamics and analyze how the two dynamical regimes connect with one another.

The work is organized as follows: In section 2 we nondimensionalize the governing Lagrangian equations and simplify them by substituting the pseudo angular momentum for the zonal velocity. The simplified system is analyzed in section 3 and the work concludes with a discussion in section 4.

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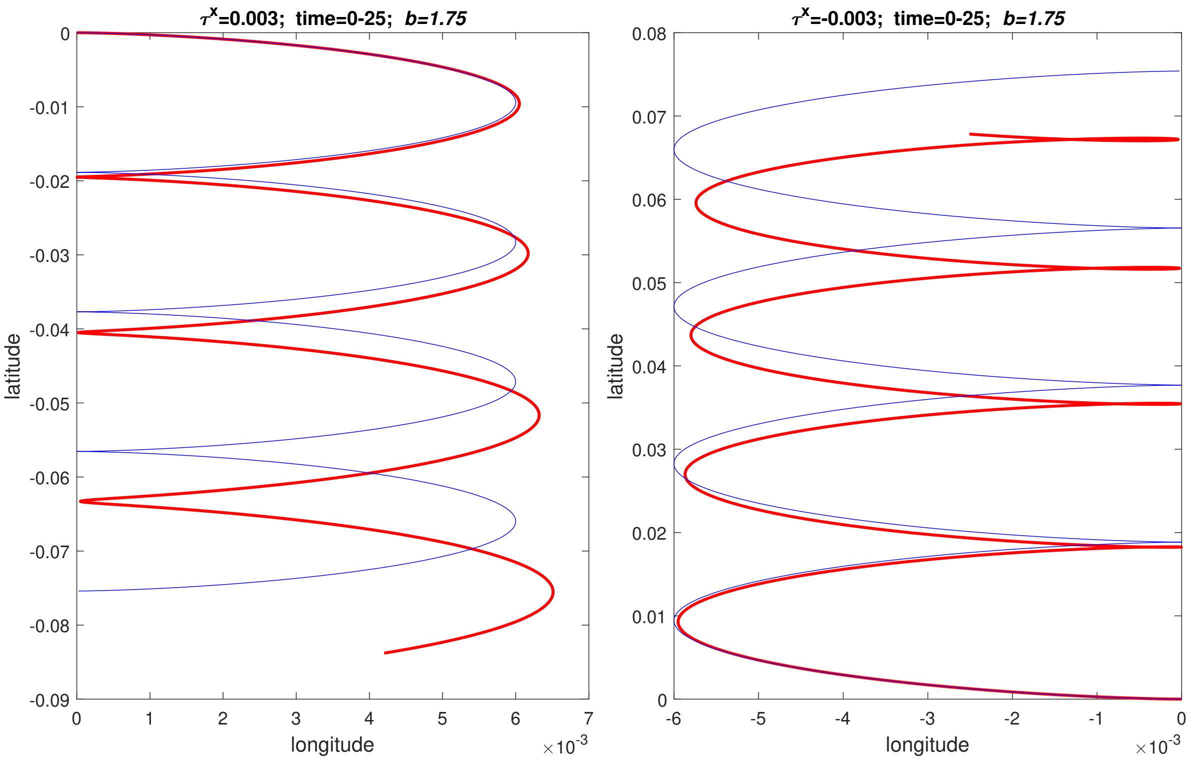

Figure 1. The (longitude, latitude) trajectories of water columns at the ocean surface subject to westward directed (right panel) and eastward directed (left panel) wind stress on the f-plane (blue curves) and on the β-plane (red curves). The time unit is the inverse of the mean Coriolis frequency and the longitude and latitude distances are scaled on Earth's radius. The value of b (scaled β) corresponds to 30° latitude. The scaling of the wind stress (τ^x) is detailed in section 2. Both trajectories start from $(x,y) = (0,0)$ located at the bottom-right point in the right panel and at the upper-left point in the left panel

2 The Nondimensional Model

The time-dependent trajectory of a column of water in the surface Ekman layer forced by the overlying wind stress on the f-plane is a fundamental problem of Physical Oceanography that is fully described in most textbooks (Gill, 1982; Pedlosky, 1987; Vallis, 2017). The governing equations describing the dynamics of vertically integrated horizontal velocity components consists of the momentum equations in the zonal and meridional directions and the (trivial) relations between these velocity components and the coordinate changes of the moving column in these directions are:

$$\frac{dx}{dt} = U, \quad \frac{dy}{dt} = V, \quad \frac{dU}{dt} = fV + \frac{\tau^x}{\rho}, \quad \frac{dV}{dt} = -fU. \tag{1}$$

Here τ^x is the zonally directed wind stress (which is positive/negative for eastward/westward directed wind, respectively), ρ is the water density, $f = f_0 + \beta y$ is the Coriolis parameter (where $f_0 = 2\Omega\sin(\phi_0)$, $\beta = 2\Omega\cos(\phi_0)/a$ with a and Ω – Earth's

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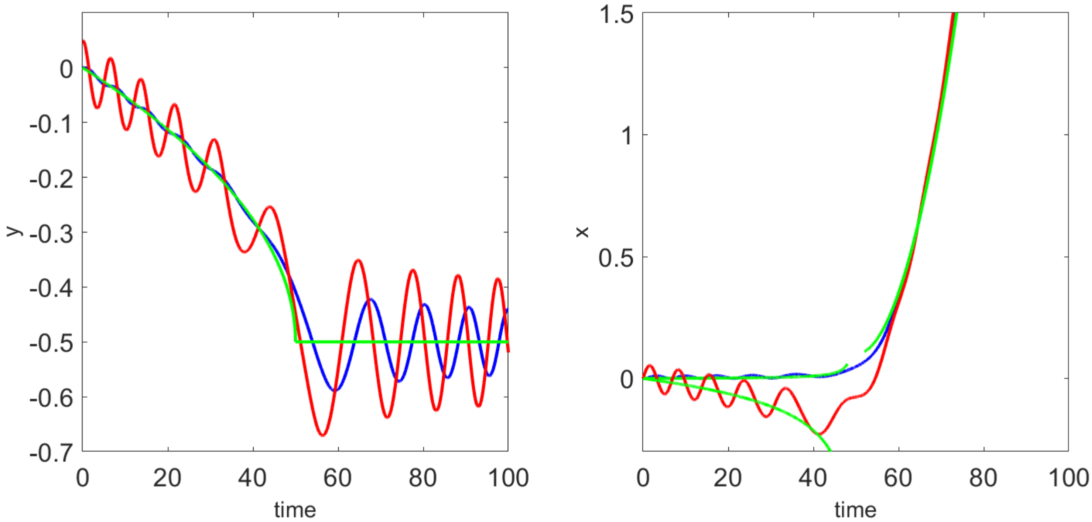

Figure 2. Numerical solutions of $y(t)$ (left panel) and $x(t)$ in system (2) - (5) starting from the fixed point $x = y = V = D = 0$ (blue curves) and from $x = V = D = 0$, but $y = 0.05$ (red curves) for $b = 2$ and $\Gamma = 0.005$. The green curves show the monotonic evolution (averaged over oscillations) of x and y described by the theory developed in section 3

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radius and rotation frequency, respectively and ϕ_0 – the latitude where the plane is tangential to Earth), U and V are the vertically integrated horizontal velocity components in the eastward and northward directions, respectively, and x and y are the respective coordinates in these directions. The only added complication of this system relative to that studied in details in e.g. chapter 9 of Gill (1982) is that here the Coriolis frequency, f, in the momentum equations is y-dependent.

The 4-dimensional system (1) can be easily integrated numerically but the general properties of its solutions can be best deciphered by reducing the number of its free parameters. This is done by scaling time, t, on $\frac{1}{f_0}$, x and y on a so the velocity scale is $f_0 a$ and the scale for the vertically integrated transport U and V is $f_0 a H$ where H is the depth (thickness) of the Ekman layer. With this scaling the nondimensional Coriolis frequency is $1 + by$ where $b = \frac{\beta a}{f_0} = \cot(\phi_0)$ is the nondimensional β. The system is further simplified by replacing U by the pseudo angular momentum, defined as $D = U - y(1 + \frac{b}{2}y)$ in nondimensional units. As was shown by Paldor (2007) when $\tau^x = 0$ i.e., in the Inertial case, D is conserved. We note that in spherical coordinates the angular momentum, which is the spherical counterpart of D, relates the zonal velocity to the latitude (Paldor, 2001). Formally, a similar quantity relating the zonal velocity (e.g. U) and the meridional coordinate, y, can also be derived in Cartesian coordinates but, unlike spherical coordinates, this conserved quantity is not the angular momentum. With these changes system (1) transforms to:

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$$\frac{dx}{dt} = D + y(1+\frac{b}{2}y), \tag{2}$$

$$\frac{dy}{dt} = V, \tag{3}$$

$$\frac{dD}{dt} = \Gamma, \tag{4}$$

$$\frac{dV}{dt} = -(1+by)(D+y(1+\frac{b}{2}y)). \tag{5}$$

Here t, x, y and V denote the nondimensional counterparts of the dimensional variables denoted by the same symbols in system (1) and, as explained above, $D = U - y(1+\frac{b}{2}y)$ is the nondimensional pseudo angular momentum. Equation (4) confirms that D is indeed conserved when $\Gamma = 0$. The solutions of this system are determined by the 4 required initial conditions and the 2 parameters: $b = \frac{\beta a}{f_0} = \cot(\phi_0)$ – the nondimensional β and $\Gamma = \frac{\tau^x}{\rho f_0^2 a H}$ – the constant, nondimensional, surface wind stress. The value of b at 30° latitude is 1.75 and for realistic values of $\tau^x/\rho \approx 2\times10^{-4} m^2 s^{-2}$, $f_0 = 10^{-4} 1/s$ and $H = 30m$ $\Gamma = 10^{-3}$ so the theory should be applicable to b of (1) and $\Gamma \ll 1$. The sign of Γ is that of τ^x – positive for eastward directed tress and negative for westward directed stress.

We conclude this section by presenting the results of numerical solutions of our system, when starting in the vicinity of the fixed point $x = 0 = y = V = D$. Figure 2 shows the time evolution of latitude y (left panel) and longitude x (right panel) for $b = 2$, $\Gamma = 0.005$ and two sets of initial conditions: $x = y = V = D = 0$ (blue lines in figure 2) and $x = V = D = 0, y = 0.05$ (red lines in this figure). The figure shows that in both cases the evolution changes qualitatively at the critical time $t = t_{cr} = (2b\Gamma)^{-1} = 50$ (at this time the nondimensional Coriolis frequency $1+by$ vanishes, see also the discussion in the next section). For $t < t_{cr}$ the evolution of y (left panel) includes a southward directed mean flow (as on the f-plane when the wind stress is directed eastward) which occurs in both trajectories while the westward drift in x ($x < 0$) occurs only when the trajectory emanates from $y \neq 0$. At $t < t_{cr}$ all trajectories include oscillations of different amplitudes that compound the mean flow (in y) and the drift (in x). In contrast, for $t > t_{cr}$, there is no mean flow in y while x grow monotonically. Furthermore, both the amplitude of the oscillations and the drift velocity of x at $t < t_{cr}$ increase as one starts out slightly off the fixed point $x = 0 = y = V = D$. We find that the evolution in figure 2 is typical to the dynamical system (2) - (5) and turn now to analyzing the dynamics that underlie the different types of evolution.

3 Analysis

The analysis of system (2)-(5) begins with the (V, y) subsystem, i.e. equations (5) and (3) along with the (trivial) solution $D = \Gamma t$ of (4). The derived solution of $y(t)$ will then be substituted in Eq. (2) to yield the zonal propagation speed. First, we combine Eqs. (5) and (3) into a single equation

$$\frac{d^2y}{dt^2} = -(1+by)\left[D + y(1+\frac{b}{2}y)\right]. \tag{6}$$

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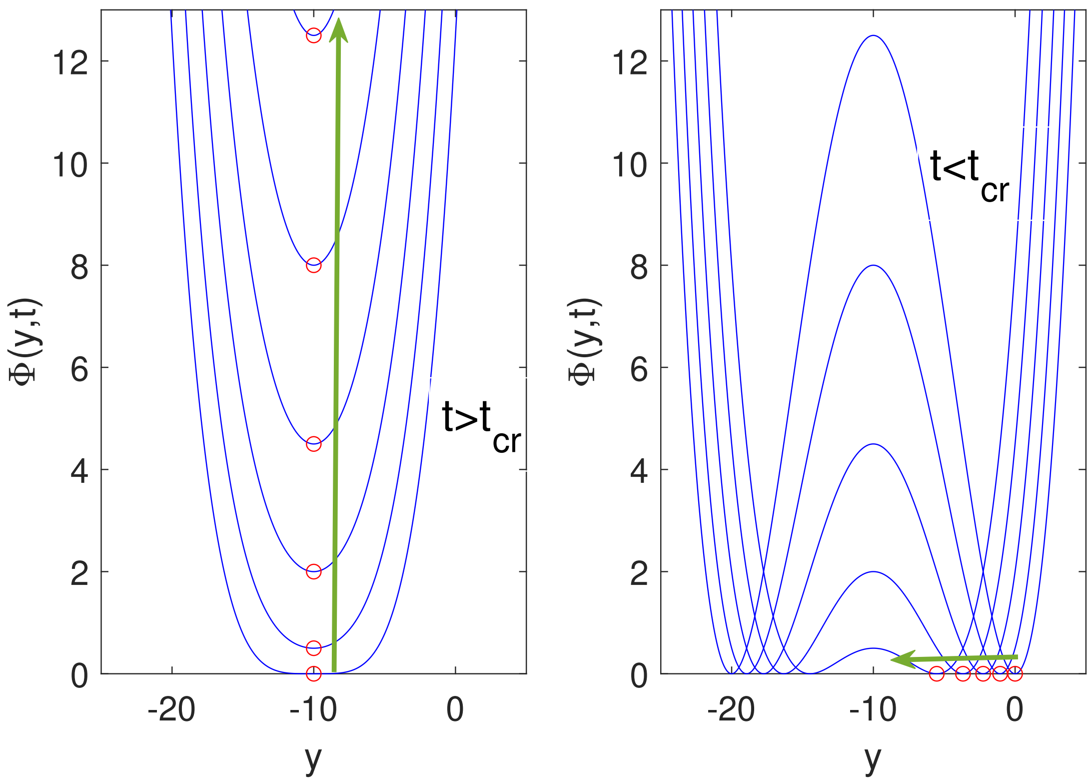

Figure 3. The change in the potential $\Phi(y,t)$ for $b = 0.1$ and $\Gamma = 0.1$ at $t = 0, 10, 20, ..., 100$. The direction of increase in time is indicated by the green arrows for $t < t_{cr}$ (right panel) and $t > t_{cr}$ (left panel). The minima, y_m^+, of the potentials are indicted by red circles

We will discuss solutions of this equation for initial conditions in the vicinity of $y = V = 0$ and assuming that Γ is sufficiently small [for the smallness condition see equation (A3) in the Appendix]. We proceed by rewriting Eq. (6) as

$$\frac{d^2y}{dt^2} = -\frac{\partial\Phi(y,t)}{\partial y} \tag{7}$$

where

$$\Phi(y,t) = \frac{1}{2}\left[\Gamma t + y\left(1 + \frac{1}{2}by\right)\right]^2. \tag{8}$$

Equation (7) describes the dynamics of a quasi-particle in a slowly (for small Γ) time varying quasi-potential well $\Phi(y,t)$. We illustrate this potential for $\Gamma = b = 0.1$ at times t = 0, 10, 20, ..., 100 in figure 3. The minima of these potentials, denoted collectively by y_m, are given by the 3 roots of:

$$\left.\frac{\partial\Phi}{\partial y}\right|_{y=y_m} = \left[\Gamma t + y_m\left(1 + \frac{1}{2}by_m\right)\right](1 + by_m) = 0. \tag{9}$$

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Two cases should be considered depending on time being below or above the critical time (discussed above)

$$t_{cr} = \frac{1}{2b\Gamma}. \tag{10}$$

For $t < t_{cr}$, there exist two minima defined by $\Gamma t + y_m \left(1 + \frac{1}{2} b y_m\right) = 0$, i.e.,

$$y_m^{\pm} = \frac{1}{b}(-1 \pm \sqrt{1 - 2b\Gamma t}) \tag{11}$$

while for $t > t_{cr}$, there exists a single minimum at

$$y_m^0 = -\frac{1}{b}. \tag{12}$$

The direction of evolution of y_m^+ in time and and its transformation into y_m^0 (red circles in figure 3) is indicated by the green arrows for $t < t_{cr}$ (right panel) and $t > t_{cr}$ (left panel) in figure 3.

The main idea of the following analysis is that since the system starts *near* $y = V = 0$, i.e. near the minimum y_m^+ of the potential, by the adiabatic theory (see pages 531-535 in Goldstein, 1980) it will *always stay near* this minimum for $t < t_{cr}$. At $t = t_{cr}$, y_m^+ transforms into y_m^0 and therefore at all $t > t_{cr}$, the system remains near y_m^0. Thus, we remain near the minimum of U at all times, while this minimum is a slowly decreasing for $t < t_{cr}$ and stays constant at $t > t_{cr}$. Since we start near the minimum y_m^+ and since for small Γ the variation of the potential is slow according to Eq. (9), we expect the solution for y to be of the form

$$y = y_m(t) + \delta y \tag{13}$$

where $y_m(t)$ starts at y_m^+ and later (i.e. at $t = t_{cr}$) transforms into y_m^0, while δy is a small perturbation. We substitute this form of solution into Eq. (6) and rewrite the resulting equation as

$$\frac{d^2 \delta y}{dt^2} = -F - \omega_0^2(t)\delta y - A\delta y^2 - B\delta y^3 \tag{14}$$

where $F = -d^2 y_m/dt^2$ is an homogeneous forcing term and using the definitions of y_m, the 3 remaining coefficients on the RHS of this equation are:

$$\omega_0^2(t) = \begin{cases} (1 + b y_m^+)^2 = 1 - 2\Gamma t, & t < t_{cr} \\ b\gamma t - 1/2, & t > t_{cr} \end{cases}. \tag{15}$$

$$A = \begin{cases} (3/2) b\omega_0, & t < t_{cr} \\ 0, & t > t_{cr} \end{cases} \tag{16}$$

and

$B = \frac{1}{2} b^2$.

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The second term on the RHS of Eq. (14) describes linear oscillations having slowly varying frequency $\omega_0(t)$, while the third and fourth terms represent the effect of small anharmonicity of the potential well near the minimum. Note that the term $y_m^+(t)$ in Eq. (13) describes slow monotonic variation of the latitude shown the green curves in our example in figure 2 at $t < t_{cr}$. No such variation exists at $t > t_{cr}$ since then $y_m = y_m^0 = const$. As will be shown below the nonlinear terms in (14) mostly affect the zonal drift in x.

Importantly, for constant parameters ω_0, A and B the solution of Eq. (14) can be found in textbooks (see e.g. pages 86-87 in Landau and Lifshitz, 1982) and has the form

$$\delta y = \frac{F}{\omega_0^2} + a\cos\psi - \frac{Aa^2}{2\omega_0^2} + \frac{Aa^2}{6\omega_0^2}\cos(2\psi) + O(a^3), \tag{17}$$

where $\psi = \omega t + \phi_0$, ϕ_0 takes into account initial conditions, and

$$\omega = \omega_0 + \left(\frac{3B}{8\omega_0} - \frac{5A^2}{12\omega_0^3}\right)a^2. \tag{18}$$

Therefore, δy includes harmonic oscillations of amplitude a and $O(a^2)$ corrections in δy and in the oscillation frequency ω. As is shown in the Appendix, when ω_0 is a slow function of time as in our case $[d\omega_0/dt \sim O(\Gamma)]$, the solution (17) remains the same, but ψ is replaced by $\psi = \int \omega dt + \phi_0$ and the oscillation's amplitude a becomes a slow function of time such that $\omega a^2 = const$. This completes our solution for the longitude y and we proceed to the latitude dynamics.

The dynamics of latitude x is governed by Eq. (2) which after substitution of (13) becomes

$$\frac{dx}{dt} = \left[D + y_m(1 + \frac{b}{2}y_m)\right] + (1 + by_m)\delta y + \frac{b}{2}\delta y^2. \tag{19}$$

Here again we consider two cases. For $t < t_{cr}$, $D + y_m^+(1 + \frac{b}{2}y_m^+) = 0$ and, therefore, by averaging in time (i.e. neglecting oscillatory components due to δy and using Eqs. (17) and (15) we get

$$\frac{d\langle x\rangle}{dt} = -\frac{ba^2}{2} + \frac{F}{\omega_0}. \tag{20}$$

This equation shows that the average zonal drift is a nonlinear phenomenon in terms of the amplitude of oscillations and is negative for $t < t_{cr}$ as seen in the examples in figure 2. In contrast, for $t > t_{cr}$, $y_m^0 = -1/b$, so $D + y_m^0(1 + \frac{b}{2}y_m^0) = \Gamma t - 1/b$ and therefore,

$$\frac{d\langle x\rangle}{dt} = \Gamma t - \frac{1}{2b} + \frac{1}{4}ba^2. \tag{21}$$

The average zonal drift is positive and monotonic as again seen in the examples in figure 2 and only weakly dependent on the amplitude of oscillation.

Figure 4 compares the $(x(t), y(t))$ trajectories emanating from different initial latitudes for eastward directed (solid curves) and westward directed (thin curves) wind stresses of identical magnitude. The various curves clearly demonstrate the effect of both the wind stress direction and the initial latitude. Trajectories emanating from $y = 0$) (right panel) are very similar to those on the f-plane. Until it reaches the equator (which is completely missing from the f-plane dynamics) a trajectory consists of a

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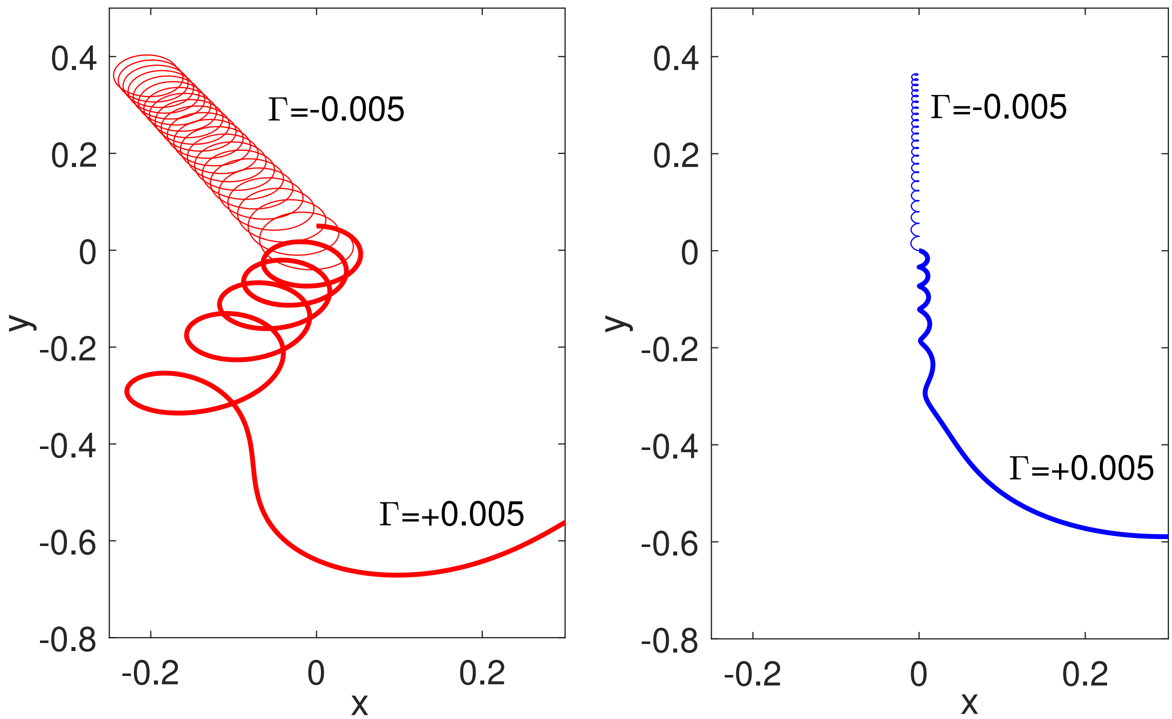

Figure 4. The different water column trajectories for $b = 2$, initial conditions $V = 0 = x = D$ and different initial latitudes. Left panel (red curves): $y(t = 0) = 0.005$; Right panel (blue curves): $y(t = 0) = 0.000$. The values of Γ are noted near each of the curves: thin curves denote negative |(westward directed) stresses and thick curves denote positive (eastward directed) stresses.

steady velocity and (inertial) oscillations. In accordance with the intuits presented in the 1 the oscillation's (inertial) frequency changes with latitude: increasing/decreasing in northward/westward directed trajectories while the oscillation's amplitude follow the opposite pattern. Trajectories similar to those shown on the left panel are also encountered when $y(t = 0) = 0$ but $V(t = 0) \neq 0$ so the difference between the trajectories on the left and right panels of figure 4 originates from the (even slight) deviation of the trajectory's origin from the fixed point y_m^+.

4 Discussion and Summary

The two simple limits of $b = 0$ (Ekman transport on the f-plane) and $\Gamma = 0$ (inertial trajectories on the β-plane) should be discussed as special cases of the involved theory presented here. These limits are well known in physical oceanography but they were never presented as limits of a single dynamical system.

In the $b = 0$ limit (wind forced transport on the f-plane) the potential in (8) becomes (recall: $\Gamma t = D$) $\Phi(y) = \frac{1}{2}(D + y)^2$. This potential has a single minimum at $y^f = -D$ and the frequency of oscillation near this point is $\omega^f = 1$. Near y^f the

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potential Φ is identical to that of Harmonic Oscillator – $\frac{1}{2}(y-y^f)^2$. The substitution $b=0$ leaves equation (4) unchanged so $D=\Gamma t$ i.e. $y^f=-D$ must decrease (or increase depending on the sign of Γ) indefinitely at the same rate as Γ. Thus the potential Φ simply translates in the $+y$ or $-y$ directions without changing its shape.

The $\Gamma=0$ limit (inertial trajectories on the β-plane) implies, according to equation (4), that D is conserved i.e. it is time-independent. Thus, system (2) - (5) has two conserved quantities – D and the energy – E. Unlike the $\Gamma\neq 0$ case studied above, with the initial conditions $U(t=0)=0=V(t=0)$ imposed here the inertial trajectory will remain indefinitely at the initial, $(x(t=0),y(t=0))$, point. With the increase in the initial energy (say by setting $V(t=0)\neq 0$), and in contrast with results derived here for $\Gamma\neq 0$, the inertial trajectory will oscillate in (V,y) while drifting westward (see Ripa, 1997; Paldor, 2007) as on the sphere (Paldor, 2001).

The solutions of nonlinear system (2) - (5) are determined by the 4 initial conditions and the values of the two parameters, β and Γ, for a total of 6 parameters! Thus, these solutions display a wide range of temporal evolution and this work clearly describes and analyzes the general properties of these solutions, illustrates in numerical examples. In particular, the mean westward drift of the trajectories shown in figure 1 does not typify the longer trajectories shown in figure 4. The sensitive dependence on parameter values (including initial conditions) is a defining property of nonlinear systems such as that studied here.

The symmetry between y_m^+ and y_m^- in the present theory suggests that for the same wind tress, Γ, the southern hemisphere's fixed point will also move towards the equator, i.e. northward. However, in all other respects the evolution near y_m^- is identical to that described above for y_m^+.

The importance of latitudes where the **curl** of the wind-stress vanishes, that play a fundamental role in (Stommel, 1948) vorticity based theory of wind driven ocean gyres, is not expected to be captured in extensions of the present Lagrangian theory. However, extensions of the present new Lagrangian theory on the β-plane can include variable wind stress, $\tau^x(y)$, which can highlight the role played by latitudes where the wind-stress **itself** vanishes. Furthermore, the extension of the present study to spherical geometry using the concepts developed here is an interesting and valuable focus of a future study.

Appendix A: Adiabatic evolution of meridional oscillations and initial conditions

In this appendix we discuss adiabatic (slow) evolution of linear longitudinal oscillations described by [see Eq. (14)]

$$\frac{d^2\delta y}{dt^2}=-\omega_0^2(t)\delta y+\frac{F}{\omega_0^2} \tag{A1}$$

and seek solution of this equation of form

$$\delta y=a(t)\cos(\int\limits_0^t\omega_0(t)dt+\phi_0)+\frac{F}{\omega_0^2}. \tag{A2}$$

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Here ϕ_0 is added to take into account initial conditions and we assume that the change of ω_0 during one period $2\pi/\omega_0$ of oscillations is small, i.e.

$$\frac{d\omega_0}{dt}\frac{2\pi}{\omega_0} \ll \omega_0 \tag{A3}$$

which is guaranteed if Γ is sufficiently small. This is our adiabaticity criterion. A similar condition, $\frac{da}{dt}\frac{2\pi}{\omega_0} \ll a$, is also assumed for the amplitude of oscillations. Next, we substitute (A2) into (A1) and neglect d^2a/dt^2 to get

$$2\frac{da}{dt}\omega_0 + a\frac{d\omega_0}{dt} = 0, \tag{A4}$$

yielding

$$\omega_0 a^2 = I = Const. \tag{A5}$$

The constant I (the action) is given by initial conditions. When the nonlinear terms in (14) are included in the analysis, all the derivation of weakly nonlinear solution as described in pages 86-87 of Landau and Lifshitz (1982) is not affected by the replacement of the linear component $a\cos(\omega t + \phi_0)$ by $a(t)\cos(\int_0^t \omega(t)dt + \phi_0)$ in the adiabatic problem which is the basis of solution (17) in section 3.

Finally, the action I, which remains constant all all times, can be calculated from the initial conditions, $\delta y(0)$ and $\delta V(0) = d(\delta y)/dt|_{t=0}$. Using (A2) we have $\delta y(0) = a(0)\cos\phi_0 + F(0)/\omega_0^2(0)$ and $\delta V(0) = -a(0)\omega_0(0)\sin\phi_0$. Then

$$a(0)^2 = \left(\delta y(0) - \frac{F(0)}{\omega_0^2(0)}\right)^2 + \left(\frac{\delta V(0)}{\omega_0(0)}\right)^2 \tag{A6}$$

and

$$I = \omega_0(0)\left(\delta y(0) - \frac{F(0)}{\omega_0^2(0)}\right)^2 + \frac{\delta V(0)^2}{\omega_0(0)}. \tag{A7}$$

The case depicted in figure 2 has $\delta V(0) = 0$, so one gets $a(0) = \delta y(0) - \frac{F(0)}{\omega_0^2(0)}$ and $I = \omega_0(0)a(0)^2$. A different scenario results for a trajectory starting with $\delta y(0) = \delta v(0) = 0$ in which case Eq. (A6) yields $a(0) = -\frac{F(0)}{\omega_0^2(0)}$.

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