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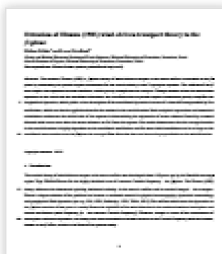

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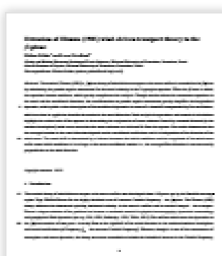

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Extension of Ekman (1905) wind-driven transport theory to the
β-plane
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Abstract. The seminal, Ekman (1905)'s, f -plane theory of wind driven transport at the ocean surface is extended to the ~~β-plane~~ β-plane by substituting the pseudo angular momentum for the zonal velocity in the Lagrangian equation. ~~The addition of the β term implies that~~ When the β term is added, the equations become nonlinear, which greatly complicates the analysis. Though rotation relates the momentum equations in the zonal and the meridional directions, the transformation to pseudo angular momentum greatly simplifies the ~~longitudinal~~ longitudinal dynamics, which yields a clear description of the meridional dynamics in terms of a slow drift compounded by fast oscillations, which can then be applied to describe the motion in the zonal direction. Both analytical expressions and numerical calculations ~~underscore~~ highlight the critical role of the equator in determining the trajectories of water columns forced by eastward directed (in the northern hemisphere) wind stress even when the water columns are initiated far from the equator. Our results demonstrate that the averaged motion in the zonal direction ~~is highly dependent on the meridional oscillations and for some initial conditions can be~~ depends on the meridional oscillations and it is independent of the direction of the wind stress. The zonal drift is determined by a balance between the initial conditions and the magnitude of the wind stress so for some initial conditions it is as large as the ~~meridional mean motion so on the β-plane the averaged flow direction does not have to be~~ mean meridional motion i.e., the averaged flow direction is not necessarily perpendicular to the wind direction.

1 Introduction

The seminal theory of wind driven transport at the ocean surface was developed about 120 years ago by the Swedish oceanographer Vagn Walfrid Ekman for the highly idealized case of constant Coriolis frequency – the f -plane. The Ekman (1905) theory addresses the downward spiraling horizontal velocity in the ocean's surface and its vertical integral – the transport. Ekman's elegant solution of the problem has become a textbook material in physical oceanography, dynamical meteorology and geophysical fluid dynamics (see e.g. Gill, 1982; Pedlosky, 1987; Vallis, 2017). For uniform wind stress the dynamics on

the f -plane consists of two parts: A steady flow to the right/left of the wind direction in the northern/southern hemisphere and inertial oscillations ~~(with~~ (of frequency f0 – the constant Coriolis frequency). However, though it is one of the cornerstones of atmosphere and ocean dynamics, the theory was never extended to include ~~increase in the Coriolis frequency with the~~ the latitudinal increase in the Coriolis frequency,

1

known as the β effect, which is the focus of the present study. ~~1~~

In contrast ~~with results derived here for F ≠ 0,~~ to the β-plane, in spherical coordinates the theory of wind-driven transport was studied numerically in Constantin and Johnson (2019) and Paldor (2002) but due to the complexity of the governing equations in these coordinates, the numerical solutions have not yielded analytic understanding. With the wind-driven dynamics on the f -plane fully understood and quantified, the β-plane offers an in-between set-up where analytical insight can complement the numerical solutions.

For given wind-stress forcing the known general differences between the dynamics on the f -plane and β-plane suggest heuristically that the extension of Ekman’s transport theory to the β-plane should include the following qualitative elements:

1. An increase/decrease in mean meridional velocity for an eastward/westward directed stress due to the decrease/increase in Coriolis frequency when the water column moves southward/northward.

2. The frequency of oscillation about the mean velocity should decrease/increase ~~(and the oscillation period should in-crease/decrease) due to the decrease/increase in Coriolis frequency along the trajectory (for an eastward directed stress while the opposite changes for westward directed tress).~~

(so oscillation period should increase/de-crease) due to the decrease/increase in Coriolis frequency along the trajectory (for an

eastward directed stress in the
northern hemisphere while the opposite changes occur for westward directed tress and
~~negative for westward directed stress.~~

~~We conclude this section by presenting the results of numerical solutions of our system, when starting in the vicinity of~~
~~the fixed point x= 0 = y = V =D. Figure 2 shows the time evolution of latitude y (left panel) and longitude x (right panel)~~
~~for b= 2, Γ = 0.005 and two sets of initial conditions: x= y = V =D = 0 (blue lines~~ in the southern hemisphere).

35 3. Since the oscillation's frequency and amplitude are inversely correlated (energy flux is unchanged) a decrease in fre-
quency should lead to an increase in amplitude and vise versa.

4. Since inertial oscillations, that form a perfectly circular motion on the f -plane, drift westward on the β-plane the averaged
zonal motion should drift to the west. A heuristic reasoning of the westward drift in terms of the change in the radius of
the inertia circle was proposed by Von Arx (1964) and complete quantitative theories of the drift were developed in Ripa

40 (1997) and Paldor (2007).

The numerical solutions of the governing Lagrangian equations (see section 2 below) shown in figure 1 fully confirm the first
3 expectations listed above but contradict the fourth one – for both westward (right panel) and eastward (left panel) stresses,
the trajectories drift to the ~~west.~~ east. From the particular example shown in figure 1 it is unclear whether the eastward transition is
a general feature of the wind driven dynamics on the β-plane or a specific occurrence related to the particular choice of initial

~~40~~ 45 ~~in a latitude where~~ conditions and/or parameter values.
In addition to resolving the issue of the zonal drift and quantifying the various rates of ~~changes the present study also~~

~~addresses the following~~ changes, the present study also

addresses the equatorial problem that exists only on the β-plane. This equatorial issue can be described as follows: An eastward
directed stress in the northern hemisphere forces a net southward directed mean flow which, on the β-plane, is accompanied
by ~~an indefinite~~ a decrease in the Coriolis frequency. ~~At some time the wind forced water column must find itself~~

Thus, at some time the wind forced water column must find itself in a latitude where

the Coriolis frequency vanishes – the equator. From that point onward the water column is subject to ~~non-rotating~~ non-rotating dynamics and must move eastward at an accelerated velocity. In the rest of this work we will estimate the time it takes the water column to change its dynamics ~~qualitatively to a non-rotating dynamics~~ from rotating to non-rotating and analyze how the two dynamical regimes connect with one another.

The work is organized as follows: In section 2 we nondimensionalize the governing Lagrangian equations and simplify them by substituting the pseudo angular momentum for the zonal velocity. The simplified system is analyzed in section 3 and the

~~1987; Vallis, 2017). The governing equations describing the dynamics of vertically integrated horizontal velocity components consists~~ work concludes with a discussion and summary in section 4.

2

x x
=0.003; time=0-25; b=1.75 =-0.003; time=0-25; b=1.75

0 0.08

-0.01 0.07

-0.02
0.06

-0.03
0.05

-0.04

0.04

-0.05

0.03
-0.06

0.02

-0.07

-0.08 0.01

-0.09 0
0 1 2 3 4 5 6 7 2 4 6 8 -6 -5 -4 -3 -2 -1 0

longitude -3 longitude -3
10 10

10-3 longitude 10-3

Figure 1. The (longitude, latitude) trajectories of water columns at the ocean surface subject to westward directed (right panel) and eastward
directed (left panel) wind stress on the f -plane (blue curves) and on the β-plane (red curves). The time unit is the inverse of the mean Coriolis
frequency and the longitude and latitude distances are scaled on Earth's radius. The value of b (scaled β) corresponds to 30◦ latitude. The
scaling of the wind stress (τx) is detailed in section 2. Both trajectories start from (x,y) = (0,0) located at the bottom-right point in the right
panel and at the upper-left point in the left panel

2 The Nondimensional Model

The time-dependent trajectory of a column of water in the surface Ekman layer forced by the overlying uniform wind stress on
the f -plane is a fundamental problem of Physical Oceanography that is fully described in most textbooks (Gill, 1982; Pedlosky,
1987; Vallis, 2017). The governing Lagrangian equations that describe the dynamics of vertically integrated horizontal velocity

60 components consist of the momentum equations in the zonal and meridional directions and the (trivial) relations between these
velocity components and the changes in the coordinate changes of the moving column in these directions are:
i.e.:
dx dy dU τx dV

= U, = V, = fV + , =−fU. (1)
dt dt dt ρ dt

Here τx is the uniform zonally directed wind stress (which is positive/negative for eastward/westward directed wind, respectively),
60 respec-
tively), ρ is the water density, f = f0 +βy is the Coriolis parameter (where f0 = 2Ωsin(ϕ0), β = 2Ωcos(ϕ0)/a with a and Ω −

~~Earth's~~

~~3~~

~~latitude~~

~~latitude~~

~~Figure 2. Numerical solutions of y(t) (left panel) and x(t) in system (2) - (5) starting from the fixed point x= y = V =D = 0 (blue curves)~~
~~and from x= V =D = 0, but y~~ 65 Earth's radius and rotation frequency, respectively and ϕ0 – the latitude where the plane is tangential to Earth), U and V are
the vertically integrated horizontal velocity components in the eastward and northward directions, respectively, and x and y are
the respective coordinates in these directions. The only added complication of this system relative to that studied in details in
e.g. chapter 9 of Gill (1982) is that here the Coriolis frequency, f , in the momentum equations is y-dependent.

3

latitude

latitude

The 4-dimensional system (1) can be easily integrated numerically but the general properties of its solutions can be best
70 deciphered by reducing the number of its free parameters. This is done by scaling time, t, on 1

f , x and y on a so the velocity
0

scale is f0a and the scale for the vertically integrated transport U and V is f0aH where H is the depth (thickness) of the
Ekman layer. With this scaling the nondimensional Coriolis frequency is 1+by where b= βa

f = cot(ϕ0) is the nondimensional
0

β. The system is further simplified by replacing U by the pseudo angular momentum, defined

as D = U − y(1+ b
2y) in

nondimensional units. As was shown by Paldor (2007) when $\tau_x = 0$ i.e., in the Inertial case, D is conserved. We note that in
~~spherical coordinates the~~ spherical coordinates the conservation of angular momentum, which is the spherical counterpart of D, ~~relates the zonal velocity to~~ yields a simple relation

between the zonal velocity and the latitude (Paldor, 2001). Formally, a similar quantity relating the zonal ~~velocity (e.g. U)~~ velocity, U , and the
meridional coordinate, y, can also be derived in Cartesian coordinates but, unlike spherical coordinates, this conserved quantity
is not the angular momentum. With these changes system (1) transforms to:

~~4~~

~~dx b~~
dx b
= D+ y(1+ y), (2)

dt 2
dy

= V, (3)
dt
dD

= Γ, (4)
dt
dV

~~b~~

~~= −(1+~~ = − b
(1+ by)(D+ y(1+ y)). (5)

dt 2

Here t, x, y and V denote the nondimensional counterparts of the dimensional variables denoted by the same symbols
in system (1) and, as explained above, D = U − y(1+ b

2y) is the nondimensional pseudo angular momentum. Equation (4)
confirms thatD is indeed conserved when Γ = 0. The solutions of this system are

determined by the 4 required initial conditions

and the 2 parameters: b= βa
f = cot(ϕ0) – the nondimensional β and Γ = τx

ρf2 – the constant, nondimensional, surface wind
0 0aH

stress. The value of b at 30◦ latitude ϕ0 = 30◦ is 1.75 and for realistic values of τx/ρ≈ 2× 10−4m2s−2, f0 = 10−41/s 10−4s−1 and H = 30m

Γ = 10−3 so the theory should be applicable to b of (1) O(1) and Γ≪ 1. The sign of Γ is that of τx – positive for eastward
directed stress and negative for westward directed stress.

90 We solve this system by starting at the origin from the fixed point y+m.

of the β-plane, i.e. x(0) = y(0) = 0 and assume that the initial V (0) andD(0) =

U(0) are sufficiently small. The numerical solutions presented below are initiated with D(0) = 0 and V (0) /= 0. However, the
definition of D implies that trajectories emanating from D(0) /= 0 and y(0) = 0 can also be calculated starting from D(0) = 0

and a suitable y(0) /= 0. Note that the choice D(0) = 0 does not restrict the generality of our solutions since the shift of time
from t to t′ = t+D(0)/Γ yields D(t′ = 0) = 0 so D(0) = 0 can be assumed. The analysis of the solutions of system (2) – (5)

95 both the amplitude of the oscillations and the drift velocity of x at t < tcr increase as one starts out including numerical examples are presented in the next section). For t < tcr the evolution of y (left panel) includes a southward directed mean flow (as on the f -plane when the
wind stress is directed eastward) which occurs in both trajectories while the westward drift in x (x < 0) occurs only when the
trajectory emanates from y /= 0. At t < tcr all trajectories include oscillations of different amplitudes that compound the mean
flow (in y) and the drift (in x). In contrast, for t > tcr, there is no mean flow in y while x grow monotonically. Furthermore,

section.

3 Analysis

The analysis of system (2)-(5) begins with the (V , y) subsystem, i.e. equations (5) and (3) along with the (trivial) solution

100 (3) and (5) along with the (trivial) solution
D = Γt of (4). The derived solution of y(t) will then be substituted in Eq. (2) to yield the zonal propagation speed. First, we

4

12 12

t10 10

8 8

6 6 t>t
cr

4 4

2 2

0 0
-20 -10 0 -20 -10 0

y y

Figure 3. -25 -20 -15 -10 -5 0 5 -25 -20 -15 -10 -5 0 5

y y

Figure 2. The change in the potential Φ(y,t) for b= 0.1 and Γ = 0.1 at t= 0,10,20, ...,100. The direction of increase in time is indicated
by the green arrows for t < tcr (right panel) and t > tcr (left (left panel) and t > tcr (right panel). The minima, y+

m, of the potentials are indicted by red circles

We will discuss solutions of this equation for initial conditions in the vicinity of y = V = 0 and assuming combine Eqs. (5) a[nd (3) into a sin]gle equation

d2y b
equations[(3) and (5) to th]e single second-order equation

d2y b
100 =−(1+ by) D+ y(1+ y) . (6)

dt2 2

5

12 12

10 10 We will discuss solutions of this equation for initial conditions in the vicinity of y = dy/dt= V = 0 and assume that Γ is
sufficiently small [for the smallness condition see equation (A3) in the Appendix]. We proceed by rewriting Eq. (6) as

d2y
=−∂Φ(y,t)105 =−∂Φ(y,t) (7)

dt2 ∂y

where [( )]2
1 1

105 Φ(y,t) = Γt+ y 1+ by . (8)
2 2

Equation (7) describes the dynamics of a quasi-particle in a slowly (for small Γ) time varying quasi-potential well Φ(y,t).
We In figure 2 we illustrate this potential for Γ = b= 0.1 at times t = 0, 10, 20, ..., 100 in figure 3. 100. The minima of these potentials, denoted
colle||ctively by ym, are given by the 3 roots of:
∂Φ ||]
1

= Γt
∂y | + |
∂y |

[( )]
1

= Γt+ ym 1+ bym (1+ bym) = 0. (9)
y=y 2

m

6

~~(y,t)~~

~~(y,t)~~

110 ~~colle|ctively b[y ym, are(given by the)3~~ Two cases should be considered depending on time being below or above the critical time
~~(discussed above)~~

~~1~~
~~tcr = . (~~ 1

tcr = . (10)
2bΓ

5

(y,t)

(y,t)

( )
For t < tcr, there exist two minima defined by Γt+ ym 1+ 1

2bym = 0, i.e.,

y±
1 √

m = (−1± 1− 2bΓt) (11)
b

while for t > tcr, there exists a single minimum ~~at~~

located at

115 ~~y±m~~ y0m =−1
. (12)
b

The direction of evolution of y+m in time and ~~and its transformation into y0m (red circles in figure 3) is indicated by the green~~
~~arrows for t < tcr (right panel) and t > tcr (left panel) in figure 3.~~

120 its transformation to y0m (red circles in figure 2) and x= V =D are indicated by the green arrows
for t < tcr (left panel) and t > tcr (right panel) in figure 2.

The main idea of the following analysis is that since the system starts near y = V = 0, i.e. near the minimum y+m of the
potential, dy/dt= V = 0, i.e. near the minimum of
the potential, y+m, by the adiabatic theory (see pages 531-535 in Goldstein, 1980) it will always stay near this minimum for t < tcr.

At
t= t 0

cr, y+m transforms into ym 120 At t= tcr, y+m transforms into y0m and therefore at all t > tcr, the system remains near y0m. Thus, we remain near the minimum
of U at all times, while this minimum is a the column remains near
the minimum of Φ at all times, while this minimum is slowly decreasing for t < tcr and stays constant at t > tcr. Since we start near the
minimum y+m and since for small Γ the variation of the potential is slow according to Eq. (9), for t > tcr. Since the
trajectory originates from the (even slight)

near the minimum y+m and since for small Γ the variation of the potential is slow (see Eq. (9)), we expect
the solution for y to be of the form

y = ym(t)+ δy (13)

125 where ym(t) starts at y+m and later (i.e. at t= tcr) transforms into y0m, while y0m and δy is a small perturbation. We substitute this form of
solution into Eq. (6) and rewrite the resulting equation as

d2δy
=−F −ω2(t)δy−Aδy2 = F −ω2

dt 0(t)δy−Aδy2 −Bδy3 (14)
dt2 0

2

where F =−d2ym/dt2 is an homogeneous forcing term and using the definitions of ym, the 3 remaining inhomogeneous forcing term and the coefficients on the
RHS of t his equation are:

−(1+ by+ 2

m) = 1− 2Γt, t < tcr

of the other 3 terms on the RHS of this equation
are:
ω2
0(t) = . (15)

byt− (1+ by+m)2 = 1− 2bΓt, t < tcr

130 = (15)
bΓt− 1/2, t > tcr

(3/2)bω0, t < tcr
A= (16)

0, t > tcr

and
B = 1 2

2b .
135 and
B = 1 2

2b .

7

In the present model, equation (11) implies d2y+m/dt

2 =−bΓ2(1−2bΓt)−3/2 so F =−d2y+m/dt2 > 0 for t < tcr. The second
term on the RHS of Eq. (14) describes linear oscillations having slowly varying frequency ω0(t),
while the third and fourth
terms represent the effect of small anharmonicity of the potential well near the minimum.
Note that for ym = y+m the term
y+m(t) in Eq. (13) describes slow monotonic variation of the latitude shown the green curves in our example in figure 2 at by the green curves in our example in figure 2 at

6

0

-0.1

-0.2

-0.3

-0.4

-0.5

-0.6

-0.7
0 10 20 30 40 50 60 70 80 90 100

time

Figure 3. Numerical solutions of y(t) in system (2) - (5) starting from x= y = V =D = 0 (blue curves) and from x= y =D = 0, but
V = 0.05 (red curves) for b= 2 and Γ = 0.005. The green curves show the monotonic evolution (averaged
over oscillations) of x and y described by the theory developed in section 3

.

curve shows the evolution of ym(t).

t < tcr. No such variation exists at t > tcr since then ym = y0m = const. As will be shown below, the nonlinear terms in (14)
140 such variation exists at t > tcr since then ym = y0m = const. As will be shown below the nonlinear terms in (14) mostly affect the zonal drift in x.

Importantly, for constant parameters ω0, A and B the solution of Eq. (14) can be found in textbooks (see e.g. pages 86-87 in
Landau and Lifshitz, 1982) and has the form

F Aa2 Aa2
δy = 2 it has the form

F Aa2 Aa2
δy =

ω2 + acosψ− +
0 2ω2

0 6ω2 θ

cos(2ψ)+O(a3), (17)
ω0 0

where ψ =(ωt+ϕ0, ϕ0 ta)kes into account initial conditions, and

3B ω = ω0 + – 5A2
145 ω = ω0 + a2. (18)
(18

8ω0 12ω3)
0

Therefore, δy includes harmonic oscillations of amplitude a and O(a2) corrections in δy and in the oscillation frequency ω.
and oscillation frequency ω (that includes
an O(a2) correction to ω0). As is shown in the Appendix, when ∫ω0 when∫ω0 is a slow function of time as in our case [dω0/dt~O(Γ)],
the solution (17) remains the same, but ψ is replaced by ψ = ωdt+ϕ0 and the oscillation's amplitude a becomes a slow
function of time such that ωa2 = const.

150 Figure 3 shows the numerical solutions of y(t) for two slightly off the fixed point
x= 0 = y = V =D. We find that the evolution in figure 2 is typical to the dynamical system (2) - (5) and turn now to analyzing
the dynamics that underlie the different types of evolution.

different initial conditions. As predicted, both the blue curve
(column initiated with x= y = V =D = 0) and red curve (column initiated with x= y =D = 0 and V = 0.05) oscillate
about the evolution curve of ym (green curve). The results show that away from the equator (i.e. for t < tcr) the amplitude of
oscillations is larger when V ̸= 0. This completes our solution for the longitude y and we proceed to the latitude latitude, y, and we proceed to the longitudinal dynamics.

7

y

1.4

1.2

1

0.8

0.6

0.4

0.2

0

-0.2

0 10 20 30 40 50 60 70 80

time

Figure 4. Numerical solutions of x(t) in system (2) - (5) starting from the same initial conditions as in figure 3 for the blue and red curves.
The green curves show the monotonic evolution (averaged over oscillations) of x described by the theory developed here for two different
values of a(t= 0).

The[dynamics of latitude]x in the zon]al direction, x, is governed by Eq. (2) which after substitution of (13) becomes
dx b b

155 = − = D+ ym(1+ ym) + (1+ bym)δy+ δy2. (19)
dt 2 2

Here again we consider two cases. For t < tcr, D+ y+m(1+ b
2y

+
m) = 0 and, therefore, by averaging in time (i.e. neglecting

oscillatory components due to δy and using Eqs. (17) and (15) we get

d⟨x⟩ ba2 F
δy) and using Eqs. (17) and (15) we get
d⟨x⟩

=−ba
2 F
+ . (20)

dt 2 ω0

This equation shows that the average zonal drift is a nonlinear phenomenon in terms of the amplitude of oscillations and is
negative for t < tcr as seen in the examples in figure 2. In contrast, for t > t 0

cr, ym oscillation. Figure 4 Since, as
160 The average zonal drift is positive and monotonic as again seen in the examples in figure 2 and only weakly dependent on the
amplitude of was shown above, F > 0 for t < tcr, the drift is determined by the balance between ba2/2 (determined by the initial displace-

ment from the fixed point V = 0,D = 0,y = 0.05 (red lines in this figure). The figure shows that in both cases the evolution changes qualitatively at the critical time
90 t= tcr = (2bΓ)−1 = 50 (at this time the nondimensional Coriolis frequency 1+ by vanishes, see also the discussion y+m) and F (∝ Γ2/ω0). Thus, the sign (direction) of the drift is independent of the
sign of Γ. In contrast, for t > tcr, y0m =−1/b, so D+ y0m(1+ b

2y
0
m) = Γt− 1/b

and therefore,

d⟨x⟩ 1 1
= Γt− 1/(2b) and therefore,

d⟨x⟩
= Γt− 1 1

\+ ba2. (21)
dt 2b 4

Figure 4 displays numerical solutions of x(t) in system (2) - (5) starting from the same initial conditions as in figure 3 for the
165 on the f -plane. Until it blue and red curves. The green curves show the monotonic evolution (averaged over oscillations) of x described by the theory

developed here for two different values of a(t= 0). As predicted, the zonal drift on the equator is positive and monotonic.
Figure 5 compares the (x(t),y(t)) trajectories emanating from different initial latitudes velocities V for eastward directed (solid

curves) and westward directed (thin curves) wind stresses of identical magnitude. The various curves clearly demonstrate the

~~effect of~~
~~both the wind stress direction and the initial latitude. Trajectories emanating from y = 0) (right~~
8

x

0.4 0.4

=-0.005
0.2 =-0.005 0.2

0 0

-0.2 -0.2

-0.4 -0.4
=0.005 =0.005

-0.6 -0.6

-0.8 -0.8
-0.2 -0.1 0 0.1 0.2 0.3 -0.2 -0.1 0 0.1 0.2 0.3

x x

Figure 5. The different water column trajectories for b= 2, initial conditions x= y =D = 0 and different initial velocities, V . Left panel
~~(red~~
~~curves): y(t= 0) = 0.005;~~ (blue curves): V (t= 0) = 0.00; Right panel ~~(blue curves): y(t= 0) =~~
~~0.000.~~ (red curves): V (t= 0) = 0.05. The values of Γ are noted near each of the curves: thin curves
denote negative ~~|(westward~~ (westward directed) stresses and thick curves denote positive (eastward directed) stresses.

~~steady velocity and (inertial)~~ effect of both the wind stress direction and the initial velocity. Trajectories emanating with V = 0 (left panel) are very similar to
~~those~~

170 those on the f -plane until one reaches the equator (which is completely missing from the f -plane dynamics) ~~a trajectory consists of a~~

~~8~~

~~0.4 0.4~~

~~=-0.005 =-0.005~~

~~0.2 0.2~~

~~0 0~~

~~-0.2 -0.2~~

~~-0.4 -0.4~~
~~=+0.005~~

~~=+0.005~~
~~-0.6 -0.6~~

~~-0.8 -0.8~~
~~-0.2 0 0.2 -0.2 0 0.2~~

~~x x~~

~~Figure 4. The different water column trajectories for b= 2, initial conditions V = 0 = x=D and different initial latitudes.~~ where a trajectory

consists of a steady translation and oscillations. In accordance with the intuits presented in the ~~1 the oscillation's (inertial) frequency changes with latitude:~~ Introduction on the f -plane the oscillation's (inertial) frequency changes with latitude i.e. increasing/decreasing in northward/westward directed trajectories while the oscillation's amplitude ~~fol- low~~ follow the opposite pattern. ~~Trajectories similar to those shown on the left panel are also encountered when y(t= 0) = 0 but V (t= 0) /= 0 so the difference between the trajectories on the left and right panels of figure 4~~ As expected, the zonal drift of the oscillations in the right panel of figure 5 is independent of the sign of Γ as it is determined by the balance between the initial deviation ~~of the trajectory's~~ from V = 0 and Γ2.

175 4 Discussion and Summary

The two simple limits of b= 0 (Ekman transport on the f -plane) and Γ = 0 (inertial trajectories on the β-plane) should be discussed as special cases of the involved theory presented here. These limits are well known in physical oceanography but they were never presented as limits of a single dynamical system.

In the b= 0 limit (wind forced transport on the f -plane) the potential in (8) becomes Φ(y) = 1 2 (D+ ~~y)2.~~

y)2 (recall: ~~Γt=D)~~ D = Γt).

This potential has a single minimum at $y_f = -D$ and the frequency of oscillation near this point is $\omega_f = 1$. Near y_f the

~~9~~

~~y~~

~~y~~

~~potential Φ~~ potential, Φ, is identical to that of Harmonic ~~Oscillator –~~ Oscillator: 1

2 $(y - y_f)^2$. The substitution b= 0 leaves equation (4) unchanged
so D = Γt i.e. $y_f = -D$ must decrease (or increase depending on the sign of Γ) indefinitely at the same rate as Γ. Thus the
potential Φ simply translates in the +y or −y directions without changing its shape.

9

y

y

The Γ = 0 limit (inertial trajectories on the β-plane) implies, according to equation (4), that D is conserved i.e. it is time-
independent. Thus, system (2) - (5) has two conserved quantities – D and the energy – E. ~~Unlike the Γ ≠ 0 case studied above,~~
~~with the initial conditions U(t= 0) = 0 = V (t= 0) imposed here the inertial trajectory will remain indefinitely at the initial,~~
~~(x(t= 0),y(t= 0), point.~~ With the increase in the initial

energy (say by ~~setting V (t= 0) ≠ 0), and in~~ increasing V (t= 0)) the inertial trajectory will oscillate in (V,y) while drifting westward (see Ripa, 1997;
Paldor, 2007) as on the sphere (Paldor, 2001). ~~The solutions of nonlinear system (2) - (5) are determined by the 4 initial conditions~~ Equation (20) and the trajectories shown in figure 5 show that the long-term
westward drift on the β-plane when Γ ≠ 0 is slower than in the inertial, Γ = 0, case and it is independent of the sign of Γ.

The solutions of the nonlinear system (2) - (5) are determined by the 2 initial conditions (x= 0 =D can be assumed without loss of generality since x does not affect the dynamics and D can be translated in time) and the values of the two parameters,

~~β and Γ,~~ b and Γ (that represent the dimensional parameters β and τx, respectively) for a total of ~~6~~ 4 parameters! Thus, these solutions display a wide range of temporal evolution and this work ~~clearly~~ describes and analyzes the general properties of these ~~solutions,~~ solutions and illustrates them in numerical examples. In particular, the ~~mean westward drift of the trajectories shown in figure 1 does not typify the longer trajectories shown in figure 4.~~ westward drift of the trajectories can be eastward (as in figure 1 or westward (as in figure 5. The sensitive dependence on parameter values (including initial conditions) is a defining property of

nonlinear systems such as that studied here.
The symmetry between y+m and y−m in the present theory suggests that for the same wind ~~tress,~~ stress, Γ, the southern hemisphere's

fixed point will also move towards the equator, i.e. northward. However, in all other respects the evolution near y−m is identical to that described above for y+m.

The importance of latitudes where the curl of the wind-stress vanishes, that play a fundamental role in (Stommel, 1948) vorticity based theory of wind driven ocean gyres, ~~is not expected to~~ can not be captured in extensions of the present Lagrangian theory. However,

extensions of the present new Lagrangian theory on the β-plane can include variable wind stress, τx(y), which can highlight the role played by latitudes where the wind-stress itself vanishes. Furthermore, the extension of the present study to spherical geometry using the concepts developed here is an interesting and valuable focus of a future study.

Appendix A: Adiabatic evolution of meridional oscillations and initial conditions

In this appendix we discuss adiabatic (slow) evolution of linear longitudinal oscillations described by [see Eq. (14)]

d2δy ~~= − Γ~~

~~ω2~~

~~dt 0(t)δy+ A1)~~

~~ω2 (~~

~~2~~

F

=−ω2(t)δ

dt2 0 y+ (1

ω2 A)

0

and seek solution of this equation of form

∫t

F

δy = a(t)cos(ω0(t)dt+ϕ0)+

~~. (A2)~~

~~ω2~~

~~0~~

~~0~~

~~10~~

ω2 . (A2)

0

0

Here ϕ0 is added to take into account initial conditions and we assume that the change of ω0 during one period 2π/ω0 of

210 ~~0~~

oscillations is small, i.e.

dω0 2π ≪ ω0 (A3)

dt ω0

10

which is guaranteed if Γ is sufficiently small. This is our adiabaticity criterion. A similar condition, da 2π

dt ω ≪ a, is also assumed

0

for the amplitude of oscillations. Next, we substitute (A2) into (A1) and neglect d2a/dt2 to get

da ~~dω~~

dω0
2 ω0 + a = 0, (A4)
dt dt

215 yielding

ω0a
2 = I = Const. (A5)

The constant I (the action) is given by initial conditions. When the nonlinear terms in (14) are included in the analysis, all
the derivation of weakly nonlinear solution as described in page∫s 86-87 of Landau and Lifshitz (1982) is not affected by the
~~t~~
replacement of the linear component acos(ωt+ϕ0) by t

a(t)cos(ω(t)dt+ϕ in he a ~~i~~

ia
0 0) t d ~~abatic~~ batic problem which is the basis of

220 solution (17) in section 3.
Finally, the action I , which remains constant all all times, can be calculated from the initial conditions, δy(0) and ~~δV~~ V (0) =

d(δy)/dt(|t=0. Using (A2))we ha
2 (ve ~~δy(0))=~~ δy(0)) = a(0)cosϕ0 +F (0)/ω2

0(0) and ~~δV~~ V (0) =−a(0)ω0(0)sinϕ0. Then
2

a(0)2 δy(0)− F (0) ~~δV (0)~~
~~=~~

~~ω2 + (A6)~~
~~0(0) ω0(0)~~

V (0)
=

ω2 + (A6)
0(0) ω0(0)

and ( )2
F (0) δV (0)2

V 2(0)

225 Data availability. TEXT

Code and data availability. TEXT

Sample availability. TEXT

I = ω0(0) δy(0)−
ω2 + . (A7)
0(0) ω0(0)

The case depicted in figure 2 has δV (0) figures 3 and 4 has δy(0) = 0, so one gets a(0) =
δy(0)− F (0)
ω2 and I = ω0(0)a(0)

2. A different scenario results
0(0)

for a trajectory starting with δy(0) = δv(0) = 0 in which case Eq. (A6) yields a(0) =− F (0)
ω2 .

0(0)

Code availability. TEXT

a2(0) = F 2(0)+V 2(0)
ω2 and I = ω0(0)a

2(0).
0(0)

Author contributions. The research on the problem was initiated by NP, who also proposed the transformation to the pseudo angular mo-
mentum while LF proposed the application of the adiabaticity theory. Both authors contributed equally to the numerical calculations and
manuscript preparation.

230 Competing interests. The authors declare that they have no conflict of interests

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Video supplement. TEXT

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