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Do phenomenological dynamical paleoclimate models have physical similarity with nature?

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Abstract. Phenomenological models may be impressive in reproducing empirical time series but this is not sufficient to claim physical similarity with nature until comparison of similarity parameters is performed. We illustrated such a process of diagnostics of physical similarity by comparing the phenomenological dynamical paleoclimate model of Leloup and Paillard (2022) with the more physically explicit Verbitsky *et al* (2018) model and established that, for the late Pleistocene, there is considerable physical similarity in terms of two crucial similarity parameters: (a) the ratio of the astronomical forcing amplitude to the terrestrial ice-sheet mass influx, and (b) the ratio of amplitudes of time-dependent positive and negative feedbacks.

1. Introduction

In the Epilogue of his monumental book "Dynamical Paleoclimatology" Barry Saltzman (2002) reflects on mathematical modeling of ultra-slow, ice-age-like paleoclimate processes and advocates for a phenomenological approach "through the construction of low-order models in which the full behavior is projected onto the dynamics of a reduced number of …highly aggregated variables…" as an alternative to a more explicit approach when a dynamical system is derived from more sophisticated models or, ideally, directly from basic laws of physics as it has been argued by Lorenz (1970). Now, 20 years later, this dispute is still far from being settled, and dynamical paleoclimate models of both a phenomenological nature (e.g., Saltzman and Maasch, 1991, Paillard, 1998, Tziperman et al, 2006, Crucifix, 2013, Leloup and Paillard, 2022) and of a more explicit type (e.g., Talento and Ganopolski, 2021) are widely used in the field without much consideration about their physical similarity or absence thereof. In this study, the author who was earlier involved in the development of phenomenological models (e.g., Saltzman and Verbitsky, 1992, 1993, 1994) but recently proposed a dynamical ice-age model derived from the conservation laws of viscous ice media (Verbitsky *et al*, 2018) joins this discussion with a basic question: *Do phenomenological dynamical paleoclimate models have physical similarity with nature?* In fluid

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dynamics, the concept of physical similarity is the cornerstone of any judgement built on model experimentations and classical similarity parameters such as the Reynolds number, the Peclet number, the Euler number, etc., describing the relative importance of different aspects of fluid flow, can be derived from fundamental conservation laws. The most basic logic of the physical-similarity approach is that such relative importance in the model must be the same as in nature. A migration from explicit conservation equations to highly integrated dynamical systems will definitely require different similarity parameters to be preserved, but this does not mean that diagnostics of physical similarity can be neglected.

To establish physical similarity, regardless of the mathematical modeling approach we are going to pursue (phenomenological or explicit), we first need to assume that there exists a natural parent dynamical system that created the time series given to us as the observations. Let us assume further that this parent system is governed by *n* physical parameters a_i such that a dependent variable of interest, *x*, can be expressed as function

$$x = \varphi(a_1, a_2, \dots, a_i, \dots, a_n) \quad (1)$$

If *k* parameters of $a_1, a_2, \dots, a_i, \dots, a_n$ are parameters with independent dimensions, then, according to π-theorem (Buckingham, 1914), in the dimensionless form, the phenomenon (1) can be described by $m = n - k$ adimensional similarity parameters $\Pi_1, \Pi_2, \dots, \Pi_i, \dots, \Pi_m$:

$$\Pi = \Phi(\Pi_1, \Pi_2, \dots, \Pi_i, \dots, \Pi_m) \quad (2)$$

Two physical phenomena have physical similarity if both of them are described in the adimensional form by the same function $\Phi(\Pi_1, \Pi_2, \dots, \Pi_i, \dots, \Pi_m)$ and have identical numerical values of similarity parameters $\Pi_1, \Pi_2, \dots, \Pi_i, \dots, \Pi_m$, though numerical values of the governing parameters $a_1, a_2, \dots, a_i, \dots, a_n$ may be different (e.g., Barenblatt, 2003).

As we have already mentioned, our knowledge about a parent dynamical system is available to us as empirical time series. It means that one of the similarity parameters, let say Π_1, is adimensional time $\frac{t}{\tau}$ (*t* and τ are dimensional time and a timescale, correspondingly), and all other parameters $\Pi_2, \dots, \Pi_i, \dots, \Pi_m$ are fixed to specific values. It means that an experimental time series can be described as

$$\Pi = \Phi\left(\frac{t}{\tau}, \Pi_2, \dots, \Pi_i, \dots, \Pi_m\right) \quad (3)$$

If we created a model dynamical system such that it is governed by *p* physical parameters b_i

$$x = \psi(b_1, b_2, \dots, b_i, \dots, b_p) \quad (4)$$

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and r parameters of $b_1, b_2, \dots, b_i, \dots, b_p$ are parameters with independent dimensions, then, again, according to π-theorem, in the dimensionless form, the model can be described by $q = p - r$ adimensional similarity parameters $\pi_1, \pi_2, \dots, \pi_i, \dots, \pi_q$:

$$\pi = \Psi(\pi_1, \pi_2, \dots, \pi_i, \dots, \pi_q) \quad (5)$$

For a specific time series, and for a fixed set of parameters $\pi_2, \dots, \pi_i, \dots, \pi_q$, the model (5) can be presented as

$$\pi = \Psi\left(\frac{t}{\tau}, \pi_2, \dots, \pi_i, \dots, \pi_q\right) \quad (6)$$

The essence of the phenomenological approach is to fit function $\Psi\left(\frac{t}{\tau}, \pi_2, \dots, \pi_i, \dots, \pi_q\right)$ to the function $\Phi\left(\frac{t}{\tau}, \Pi_2, \dots, \Pi_i, \dots, \Pi_m\right)$ under the "best" set of parameters $\pi_2, \dots, \pi_i, \dots, \pi_q$, i.e. to equate model time series $\Psi\left(\frac{t}{\tau}, \pi_2, \dots, \pi_i, \dots, \pi_q\right)$ and natural, empirical, time series $\Phi\left(\frac{t}{\tau}, \Pi_2, \dots, \Pi_i, \dots, \Pi_m\right)$:

$$\Psi\left(\frac{t}{\tau}, \pi_2, \dots, \pi_i, \dots, \pi_q\right) = \Phi\left(\frac{t}{\tau}, \Pi_2, \dots, \Pi_i, \dots, \Pi_m\right) \quad (7)$$

It is obvious that even if the goal (7) is achieved at every $\frac{t}{\tau}$ - point, we still cannot claim the model (6) to be physically similar to "nature" (3) because we cannot compare similarity parameters $\pi_2, \dots, \pi_i, \dots, \pi_q$ and $\Pi_2, \dots, \Pi_i, \dots, \Pi_m$ for one simple reason: we do not know enough about $\Pi_2, \dots, \Pi_i, \dots, \Pi_m$. Even if we are reasonably confident about the physics involved in the natural phenomena $\Phi\left(\frac{t}{\tau}, \Pi_2, \dots, \Pi_i, \dots, \Pi_m\right)$, and each similarity parameter π_i, representing a specific aspect of the phenomena, is believed to have its counterpart Π_i in the "real" dynamical system (i.e., $q = m$), we still do not know the numerical values of similarity parameters $\Pi_2, \dots, \Pi_i, \dots, \Pi_m$, and therefore we cannot judge if the π_i-physics in the model is as significant as the Π_i-physics of nature. Simply speaking, merely *matching a proposed phenomenological model with empirical data does not make a case for physical similarity* because it does not provide an evidence that it happens for the right reason, the reason being the similarity parameters of the right value, i.e., $\pi_i = \Pi_i$.

For example, it is undisputable that astronomical forcing plays an important role in forming Pleistocene climate variability. Variations of insolation (e.g., mid-July insolation at 65° N, e.g., Berger and Loutre, 1991), of the amplitude ε and dominating periods of the precession and obliquity thus create

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the similarity parameter $\Pi_2 = \frac{\varepsilon}{a}$ that is the ratio of the intensity of the astronomical forcing to the intensity of the terrestrial ice-sheet mass influx (ε is measured in units of a, where a is snow accumulation minus ablation). Practically all phenomenological ice-age models incorporate astronomical forcing and terrestrial ice-sheet mass influx and therefore a corresponding similarity parameter π_2 can always be constructed but we do not have the means to judge if $\pi_2 = \Pi_2$.

Thus, we have phenomenological models that occasionally may be successful in reproducing experimental time series, but we cannot be confident that the similarity parameters employed are fully legitimate, and, therefore, any further use of such models is rather speculative. On the other hand, most comprehensive space-resolving models are still not practical for calculating climate on Pleistocene timescales. In this situation, some insight could be found in the analysis of physical similarity of two dynamical models, the phenomenological dynamical paleoclimate model of Leloup and Paillard (2022), LP22 thereafter, that will provide us with the function Ψ and the dynamical paleoclimate model of Verbitsky *et al* (2018), VCV18 thereafter, that, indeed, cannot fully represent nature but, certainly, is more physically explicit and, therefore, will be nominated to provide us with the function Φ.

Obviously, since the time series produced by the LP22 model and by the VCV18 model are not identical, the models are not physically similar in the full sense of the equation (7). We will demonstrate though that the answer to the physical-similarity question is not that straightforward if our dependent variable of interest x is not necessarily a time series but a time-independent attribute such as the period of the system response to the astronomical forcing.

2. Method

The dimensional analysis and similarity properties of the VCV18 model have already been comprehensively described (Verbitsky and Crucifix, 2020, Verbitsky, 2022). It was demonstrated that its large-scale variability is mostly governed by two dimensionless parameters: by the ratio of the astronomical forcing amplitude ε to the terrestrial ice-sheet mass influx, $\Pi_2 = \varepsilon/a$ and by the so-called *V*-number, $\Pi_3 = V$ that is the ratio of amplitudes of time-dependent positive and negative feedbacks. For example, the period P of the VCV18 system response to the astronomical forcing of period T can be written as:

$$\frac{P}{T} = \Phi(\Pi_2, \Pi_3) \quad (8)$$

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We will now focus on the dimensional analysis of the LP22 model. It is described by two differential equations, first, for the growing ice volume,

$$\frac{dv}{dt} = -\frac{I}{\tau_i} + \frac{1}{\tau_g} \tag{9}$$

and for the waning ice volume

$$\frac{dv}{dt} = -\frac{I}{\tau_i} - \frac{v}{\tau_d} \tag{10}$$

Here, v and I are normalized ice volume and astronomical forcing, correspondingly, τ_i, τ_g, and τ_d are dimensional timescales. Additionally, if $I < I_0$ the system switches from equation (10) to equation (9), and if $I + v > V_0$, the system switches from equation (9) to equation (10).

First, we will convert equations (9) and (10) into a form more convenient for our purpose:

$$\frac{dv}{dt} = \frac{1}{\tau_g}\left(-\frac{I\tau_g}{\tau_i} + 1\right) \tag{11}$$

$$\frac{dv}{dt} = \frac{1}{\tau_g}\left(-\frac{I\tau_g}{\tau_i} - \frac{v\tau_g}{\tau_d}\right) \tag{12}$$

If (without any loss of generality) we consider astronomical forcing I as a sinusoid of amplitude ε and of period T, then dynamical properties like, for example, a period of the system response to the astronomical forcing, P, will be fully described by 6 parameters:

$$P = \psi\left(\frac{\varepsilon\tau_g}{\tau_i}, T, V_0, \tau_g, \frac{\tau_d}{\tau_g}, I_0\right) \tag{13}$$

If we choose period T as a parameter with independent dimensions, then, according to π-theorem:

$$\frac{P}{T} = \Psi\left(\frac{\varepsilon\tau_g}{\tau_i}, V_0, \frac{T}{\tau_g}, \frac{\tau_d}{\tau_g}, I_0\right) \tag{14}$$

The parameter I_0 is settled as a constant, $I_0 = 0$ (LP22), and the period P is not very sensitive to the choice of $\frac{\tau_d}{\tau_g}$ as long as $\tau_g > \tau_d$, and therefore without loss of much of physical content we can set $\frac{\tau_d}{\tau_g} = 0$ (instant disintegration of an ice sheet) such that:

$$\frac{P}{T} = \Psi\left(\frac{\varepsilon\tau_g}{\tau_i}, V_0, \frac{T}{\tau_g}\right) \tag{15}$$

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Finally, we notice that the period P is largely defined not by individual values of $V_0, \frac{T}{\tau_g}$ but by their ratio $V_0 \frac{\tau_g}{T}$ and therefore:

$$\frac{P}{T} = \Psi(\pi_2, \pi_3) \quad (16)$$

where $\pi_2 = \frac{\varepsilon\tau_g}{\tau_i}$, $\pi_3 = V_0 \frac{\tau_g}{T}$

Now, we are ready to discuss physical similarity. Both VCV18 and LP22 models are fairly successful in reproducing late-Pleistocene ~100-kyr-period variability, meaning that the first necessary condition of physical similarity is satisfied:

$$\Psi(\pi_2, \pi_3) = \Phi(\Pi_2, \Pi_3) \quad (17)$$

The physical meaning of $\pi_2 = \frac{\varepsilon\tau_g}{\tau_i}$ is the ratio of the astronomical forcing amplitude $\frac{\varepsilon}{\tau_i}$ to the terrestrial ice-sheet mass influx $\frac{1}{\tau_g}$, i.e. it is identical to the physical meaning of Π_2. Numerically, for the Late Pleistocene, in LP22, $\pi_2 \sim 3$, and, in VCV18, $\Pi_2 \sim 2$, this may be considered being reasonably close. Accordingly, we may conclude that models VCV18 and LP22 *are physically similar in terms of the similarity parameter that is the ratio of the astronomical forcing amplitude to the terrestrial ice-sheet mass influx.*

Positive and negative feedbacks are not explicitly presented in equation (9) and the similarity parameter $\pi_3 = V_0 \frac{\tau_g}{T}$ is the only implicit outcome of the interplay between positive and negative feedbacks. Yet, we may suggest with confidence that the ratio of positive-to-negative feedbacks in (9) is implicitly set to be $V = 1$, because this is the only possibility that may provide a linear growth of ice volume in (9). Thus (17) can be re-written as

$$\Psi[\pi_2, \pi_3(V)] = \Phi(\Pi_2, \Pi_3) \quad (18)$$

Here $\pi_3(V)$ means that π_3 is a function of V, i.e.:

$$\pi_3 = V_0 \frac{\tau_g}{T}, if\ V = 1 \quad (19)$$

and equation (18) can be re-written as

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$$\Psi'(\pi_2, V) = \Phi(\Pi_2, V) \quad (20)$$

In VCV18, the numerical value of $\Pi_3 = V = 0.75$ for the late Pleistocene, and, as we just discussed, in LP22, $V = 1$, i.e. close to the VCV18 value. We may therefore conclude that models VCV18 and LP22 *are physically similar in terms of the similarity parameter that is the ratio of amplitudes of time-dependent positive and negative feedbacks.* This result is remarkable because, in fact, the entire LP22 model can be described by the following powerful statement: If positive and negative feedbacks of the global ice-climate system are in balance ($V = 1$), the period of the system response is approximately equal to $V_0\tau_g$.

Importantly, the postulated $V = 1$ of LP22 is not constrained by any specific nature of positive or negative feedbacks. Likewise, the *V*-number in VCV18 is a conglomerate similarity parameter (Verbitsky, 2022) and thus it is not connected to a specific feedback's physics. This observation makes our similarity finding even more general.

It is also interesting that, since $\Phi(2, 0.75) \sim 2$, for $T = 41$ kyr, i.e., VCV18 makes period doubling of the obliquity-period forcing, and since $\Psi(3, 1) = \Phi(2, 0.75)$, we may conclude that LP22 makes obliquity-period doubling as well.

We have to emphasize however, that the *V*-number similarity between LP22 and VCV18 models can be observed only for the late Pleistocene period. The early Pleistocene variability is reproduced in the VCV18 model by $V \rightarrow 0$ (reduced impact of positive feedbacks) while LP22 has the assumption of $V = 1$ unchanged.

This newly discovered physical similarity of two very different models is not intuitive and adds credibility to both of them. For the LP22 model, it is a demonstration that the model is physically viable in terms of two critical similarity parameters, and its most questionable saw-tooth variability may be supported by some physical reasoning. For VCV18, which itself does not demonstrate exceptional performance in reproducing empirical time series, physical similarity with the model most celebrated for matching empirical data, it is an encouraging sign also. Nevertheless, even though the analysis revealed more physical similarity between the two models than one may expect, the VCV18 model is not true nature, and therefore such a comparison is by no means a final verdict on the LP22 model. Yet it illustrates the importance of the physical-similarity analysis.

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3. Conclusions

Are phenomenological dynamical paleoclimate models physically similar to nature? We demonstrated that, though they may be remarkably accurate in reproducing empirical time series, this is not sufficient to claim physical similarity with nature until comparison of similarity parameters is performed. We illustrated such a process of diagnostics of physical similarity by comparing the LP22 phenomenological dynamical paleoclimate model with the more explicit VCV18 model and established that, for the late Pleistocene, there is physical similarity in terms of two crucial similarity parameters, namely, the ratio of the astronomical forcing amplitude to the terrestrial ice-sheet mass influx and the ratio of amplitudes of time-dependent positive and negative feedbacks. We will not, indeed, have a definite answer until we learn more about "true" similarity parameters $\Pi_2, \ldots, \Pi_i, \ldots, \Pi_m$.

But how can we learn about natural similarity parameters $\Pi_2, \ldots, \Pi_i, \ldots, \Pi_m$? We think that Saltzman (2002) was right when he proposed that "the essential slow physics is to be sought in the low-order models." We suggest here that this "essential slow physics" may be an inventory of candidate similarity parameters $\pi_2, \ldots, \pi_i, \ldots, \pi_q$. Like, for example, the Reynolds number compares inertial and viscous forces, or the Peclet number speaks to the significance of advective heat transfer relative to heat diffusion, each of these candidate similarity parameters must describe relationships that are critical for dynamical paleoclimatology. At this point, we can confidently recommend into this inventory two ratios that, we believe, largely define Pleistocene climate – the ratio of intensities of orbital and terrestrial forcings and the ratio of intensities of system's positive and negative feedbacks (the *V*-number). Since our proposed similarity parameters are so general, it is difficult to imagine other similarity parameters that would not fall into these two categories, but the final say on this belongs, indeed, to the super-models. From here, we deviate from Saltzman's (2002) proposal and instead of following his idea that more explicit models should be tuned to satisfy a best phenomenological model, we propose to use available super-models for research and evaluation of natural similarity parameters $\Pi_2, \ldots, \Pi_i, \ldots, \Pi_m$. This will hopefully close a gap between phenomenological and physical models.

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