RC1 Comment

The authors make a bold claim in the manuscript that they have “extended” the 3DVAR algorithm (section 3) by including time distributed observations in the cost function, and inflating their associated observation error due to the “sampling time error”. First, this is nothing new or different than FGAT: even though innovations are calculated in time, collocated observations cannot be assimilated simultaneously. Second, analysis increments are still being computed only at the analysis time, so the corrections are not applied at the time of the observations that is different from the analysis time. This is due to the nature of 3DVAR and its static covariance. I thus don’t understand why the authors claim this is an extension of 3DVAR. The authors also liken their cost function to and consider it to be a reduced 4DVAR cost function. The similarity of the const functions should not be a basis for comparison, even via reduction, of 3DVAR and 4DVAR. Their respective cost functions are minimized under different constraints. In 4DVAR, those constraints include strong or weak model dynamics, and in 3DVAR there is no dynamical constraints. The authors should avoid making such confusing comparisons. The method proposed in section 3 is nothing but a MS3DVAR with FGAT.

Reply: We revised the manuscript in response to this comment.

3 An three-dimensional multiscale scheme

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The MSDA-SWOT system has been developed to address new challenges relating to the fine-scale resolution and data density of satellite observations in the SWOT altimetry era. As such, the numerical ocean model used should be high resolution with a grid spacing on the order of 1 km or finer. This imposes a major computational challenge to formulate MSDA-SWOT. Both the field campaign and SWOT measurements are localized to a limited area. Another major challenge is to assimilate these localized measurements seamlessly along with the broad routine observations. Otherwise, a spurious circulation surrounding the observing area may develop, and data assimilation could even fail. To address these two major challenges, we have formulated a particular MSDA scheme, which is based on a 3DVAR algorithm that aims to assimilate observations over a time window.

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3.1 3DVAR Formulation

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For a very high-resolution model, 3DVAR is a scheme often used in both meteorological and oceanic applications (e.g., Gustafsson, et al., 2018), because of its computational efficiency. Algorithmically, a 3DVAR scheme can assimilate observations taken only at an instantaneous

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time. In practice, observations over a time window are assimilated with the assumption that they are taken at the DA time. As such, three is a difference between the observation time and DA time. This time difference inputs error to the system, which we term as ‘sampling time error’. In practice, a short time window is selectively chosen as a compromise between incorporating more observational information, and keeping the sampling time error at an acceptable level.

To alleviate this short time window limitation in 3DVAR, we here present a method to assimilate observations for a longer time window by account for sampling time error. Suppose that at time t_k $(k = 1,2,\cdots,K)$, a number m of observations (note that m could vary in time) are available and placed into an m-vector $\boldsymbol{y}_k^o$. $\boldsymbol{R}_k^o$ is observational error covariance, and $\boldsymbol{R}_k^0 = \langle \boldsymbol{e}_k^0 (\boldsymbol{e}_k^0)^T \rangle$, where $\boldsymbol{e}_k^0$ is the observational error associated with $\boldsymbol{y}_k^o$. The state variable at t_0 can be denoted as an n-vector, $\boldsymbol{x}_0$.

We write the 3DVAR cost function as

$$J(\boldsymbol{x}_0) = \frac{1}{2}\left(\boldsymbol{x}_0 - \boldsymbol{x}_0^b\right)^T \boldsymbol{B}_0^{-1}\left(\boldsymbol{x}_0 - \boldsymbol{x}_0^b\right) + \frac{1}{2}\sum_{k=0}^{K}\left[H_k(\boldsymbol{x}_0) - \boldsymbol{y}_k^o\right]^T \left(\boldsymbol{R}_k^0\right)^{-1}\left[H_k(\boldsymbol{x}_0) - \boldsymbol{y}_k^o\right] \quad . \quad (1)$$

Here $\boldsymbol{B}_0$ is the background error covariance, and $\boldsymbol{B}_0 = \langle \boldsymbol{\varepsilon}_0^b \left(\boldsymbol{\varepsilon}_0^b\right)^T \rangle$, where $\boldsymbol{\varepsilon}_0^b$ is the background error associated with the background state $\boldsymbol{x}_0^b$. $\boldsymbol{H}_k$ is often known as an observation operator that maps the state variable to $\boldsymbol{y}_k^o$.

The cost function (1) has a form similar to the four-dimensional variational data assimilation (4DVAR) algorithm (e.g., Li & Navon, 2001), which has been implemented for oceanic applications (e.g., Weaver et al., 2003; Moore, et al., 2004; Zhang et al., 2010; Ndgodock & Carrier, 2014). In 4DVAR, $\boldsymbol{x}_0$ in the observation related terms in (1) is replaced by $\boldsymbol{x}_k$, in order to assimilate observations over a time window. Although the cost function (1) has a form of a 4DVAR cost function. It is fundamentally different from 4DVAR, since no forecast is applied, or only a persistence forecast model is used in 4DVAR.

The cost function (1) is mainly used to account explicitly for the observation error due to the difference between the observation and DA time. By definition, the observational error in (1) has the form (e.g., Li et al. 2015),

$$\boldsymbol{e}_k^0 = \boldsymbol{y}_k^o - H_k(\boldsymbol{x}_0^t), \qquad (2)$$

where $\boldsymbol{x}_k^t$ is the unknown true state. We can write (2) in the expansion form

$$\begin{aligned}\boldsymbol{e}_k^0 &= \left(\boldsymbol{y}_k^o - \boldsymbol{y}_k^t\right) + \left(\boldsymbol{y}_k^t - H_k(\boldsymbol{x}_k^t)\right) + \left(H_k(\boldsymbol{x}_k^t) - H_k(\boldsymbol{x}_0^t)\right) \\ &= \boldsymbol{e}_k^m + \boldsymbol{e}_k^r + \boldsymbol{e}_k^{mp}\ , \qquad (3)\end{aligned}$$

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where $\boldsymbol{y}_k^t$ is the unknown true value of $\boldsymbol{y}_k^o$.

In (3), the first term is the measurement error, and the second term the representation error due to the inaccurate observation operator. These two terms are well known. The last term is a new type of observational error. We can understand it in the following way. If all the observations are taken at t_0, this type of error does not occur. We follow the formulation (3) to estimate the error arising from the difference between the observation time and DA time.

The MSDA-SWOT system is based on (1) and (3). Because t is allowed to be negative in (1), we assimilate observations prior to and after the DA time. We note that this 3DVAR has the same objective as First Guess at Appropriate Time (FGAT), another scheme used in conjunction with 3DVAR (e.g. Martin et al., 2015), which is also discussed in Archer et al. (2021).

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